Partial differential equations
Lecture notes
Summer 2018
University of Ulm

Julian Scheuer

September 27, 2018

Contents

Chapter 1

Introduction and basic tools

1.1 Simple examples

The calculations in this section aim to give the reader first intuition and some examples. They are not always rigorous. In Section 1.2 we will start the rigorous treatment. A more detailed introduction with many examples can be found in [1, 16].

Heat- and Laplace equation

Let

$$u\colon (0,T) \times \mathbb{R} \to [0,\infty)$$

be twice differentiable. For each t think of $u(t,\cdot)$ as a hear distribution for example in a metallic rod. We want to deduce a law for the evolution of u in time heuristically. In an arbitrary interval (x,y) let $m(t,x,y)$ the total heat at t,

$$m(t,x,y) = \int_x^y u(t,z)\ dz.$$

Then

$$\frac{\partial m}{\partial t}(t,x,y) = \int_x^y \frac{\partial u}{\partial t}(t,z)\ dz.$$

It seems plausible, that the total heat in (x,y) grows proportionally to the spatial gradient at x an y,

$$\frac{\partial m}{\partial t}(t,x,y) = \frac{\partial u}{\partial z}(t,y) - \frac{\partial u}{\partial z}(t,x).$$

Hence

$$\int_x^y \frac{\partial u}{\partial t}(t,z)\ dz = \frac{\partial u}{\partial z}(t,y) - \frac{\partial u}{\partial z}(t,x)$$

and after differentiation with respect to y we get

$$\frac{\partial u}{\partial t}(t,y) = \frac{\partial^2 u}{\partial z^2}(t,y).$$

This is the one-dimensional version of the *heat equation*, that u has to satisfy due to the model assumptions. This is a simple example of a *partial differential equation*, an algebraic equation between the derivatives of the function u. A more precise definition follows later. The higher dimensional version of the heat equation for a function,

$$u\colon (0,T)\times\mathbb{R}^n \to [0,\infty),$$

$$\partial_t u = \Delta u,$$

can be derived by a similar heuristic argument. Here Δ denoted the so-called *Laplace-operator*

$$\Delta = \sum_{i=1}^n \partial^2_{x^i x^i}.$$ [1]

Hence a distribution function

$$\tilde{u}\colon \mathbb{R}^n \to [0,\infty)$$

that describes an equilibrium (constant heat everywhere) must satisfy

$$\Delta\tilde{u} = 0.$$

This is the so-called *Laplace-equation* and its solutions are called *harmonic functions*. These two equations, the Laplace- and the heat equation, are the prototypes of two classes of equations, that will play a major role in this course. They belong to the so-called *elliptic* resp. *parabolic equations*.

More generally it is possible to include independent local heat sources into our model, i.e. at some $x \in \mathbb{R}^n$ we have a heat source of intensity $f(x)$. The heat evolution is then given by

$$\partial_t u = \Delta u + f$$

and its equilibria are solutions of the *Poisson-equation*

$$\Delta\tilde{u} + f = 0.$$

Wave equation

Beside the parabolic and elliptic equations, the *wave equations* form a third improtant class of partial differential equations. For example, they model a vibrating string or membrane. The prototype has the form

$$\partial^2_{tt} u = \Delta u.$$

Again we want to motivate this equation by a heuristic. Let $I = (a,b) \subset \mathbb{R}$ be an arbitrary interval and let $u(t,x)$ be the displacement of the string with

---

[1]
$$\partial^m_{x^{i_1}\dots x^{i_m}} = \frac{\partial^m}{\partial x^{i_1}\cdots\partial x^{i_m}}.$$

respect to the x-axis. The crucial model assumption is that the force F that acts onto the mean displacement

$$A = \frac{1}{b-a} \int_a^b u$$

is given by the change of displacement at the boundary points (make a sketch to visualize the idea),

$$F = \partial_x u(t,b) - \partial_x u(t,a) = \int_a^b \partial^2_{xx} u(t,y)\ dy.$$

Newton's law says

$$(b-a)\partial^2_{tt} A = F$$

(we assume unity mass density) and hence

$$\int_a^b \partial_t^2 u(t,y)\ dy = \int_a^b \partial^2_{xx} u(t,y)\ dy.$$

This holds for all intervals (a,b) and hence

$$\partial^2_{tt} u = \partial^2_{xx} u.$$

Minimal surface equation

Until now all equations were *linear*, i.e. the corresponding differential operators

$$\Delta, \quad \partial_t - \Delta, \quad \partial^2_{tt} - \Delta$$

are linear maps on the space of twice differentiable functions. Now we consider a nonlinear example, which stems from a natural geometrical problem.

Suppose, on an open and connected set $\Omega \subset \mathbb{R}^n$ we have a function $u \in C^2(\Omega)$[2], that minimizes the surface area $\mathcal{F}(u)$ of its graph

$$G(u) = \{(x, u(x)) \colon x \in \Omega\}$$

within this class, given boundary values

$$u_{|\partial\Omega} = \varphi.$$

We use the *direct method of the calculus of variations* to deduce an equation for u. Let $\eta \in C_c^\infty(\Omega)$[3] a *test function*. Since $\eta_{|\partial\Omega} = 0$ and u minimizes the area, we have

$$\forall t \in \mathbb{R} \colon \mathcal{F}(u) \leq \mathcal{F}(u + t\eta).$$

[2] I.e. u is twice continuously differentiable

[3] η is infinitely often differentiable and

$$\operatorname{supp} \eta := \overline{\{x \in \Omega \colon \eta(x) \neq 0\}} \subset \Omega$$

The surface area of the graph is given by

$$\mathcal{F}(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}.$$

There holds

$$0 = \frac{d}{dt}\mathcal{F}(u + t\eta)_{|t=0} = \int_\Omega \frac{\langle \nabla u, \nabla \eta \rangle}{\sqrt{1 + |\nabla u|^2}} =^{4} - \int_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right)\eta.$$

This equation holds for all text functions η and by the *fundamental lemma of the calculus of variations*[5] there holds

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0.$$

This is the so-called *Minimal surface equation*, on of the mostly studied equations of the field of geometric analysis.

1.1.1 Exercise. In case $n = 1$ prove that we get the expected minimizers (what are they?).

Boundary value problems and PDE

We have seen several examples of partial differential equations and naturally ask the question of solvability, i.e. of the *existence* of a solution. On the set $\Omega \subset \mathbb{R}^n$ consider the Laplace-equation

$$\Delta u = 0.$$

It is clear, that every affinely linear map

$$u(x) = Ax + b$$

with a linear map $A\colon \mathbb{R}^n \to \mathbb{R}$ solves this equation. Hence we have existence in this case. However, there are many solutions. To get *uniqueness*, we have to impose conditions, similar to the theory of ordinary differential equations, where we have to impose initial values. In our situation we can for example impose certain values of u on the boundary $\partial\Omega$, i.e. we consider the so-called *boundary value problem*

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega \\ u &= \varphi \quad \text{on } \partial\Omega. \end{aligned}$$

A boundary value problem is also called *Dirichlet-problem*, if we prescribe the values of the solution on the boundary. Alternatively we can prescribe the normal derivatives,

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= \psi \quad \text{on } \partial\Omega \end{aligned}$$

[4] Partial integration, details later

[5] Details later

and then we call this problem *Neumann-problem*. Soon we will be able to show, that these problems possess unique solutions for certain φ and ψ.

For this purpose we will use *Hilbert space methods*. The essence of the idea is to view the Laplace-operator as a linear operator between suitable Hilbert spaces of *weakly differentiable functions*[6], which satisfies the assumptions of the Riesz representation theorem. These Hilbert spaces are so big, that it is relatively easy to find a solution. However, the spaces a re so big, that it is even not clear, if these *generalized solutions* are differentiable in the classical sense. This question will then be investigated within the *regularity theory*.

Now we give a definition and a broad classification of partial differential equations.

1.1.2 Definition. (i) Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be open and

$$F\colon W \subset \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \ldots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$$

a map on an open set W. An equation of the form

$$F(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x)^{7} = 0, \quad x \in \Omega, \tag{1.1}$$

is called *partial differential equation (PDE) of k-th order.*

(ii) (1.1) is called *quasilinear*, if F is affinely linear in the first variable.

(iii) (1.1) is called *semilinear*, if $\partial F/\partial a^{i_1 \cdots i_k}$ only depends on $x \in \Omega$.

(iv) (1.1) is called *linear*, if F is affinely linear in the first $k+1$ variables.

(v) If (1.1) does not belong to one of the above categories, it is often called *fully nonlinear*.

1.1.3 Remark. (1.1) is to be understood symbolically. Of course it is not clear yet, if this equation admits a k-times differentiable solution.

1.1.4 Exercise. Determine for the Poisson-equation, the heat equation, the wave equation and the minimal surface equation the defining F and the correct equation type (linear, semilinear etc.).

In this course we will restrict our attention to equations of second oder. This is more or less a matter of taste, whereas they certainly belong to the most important and best understood equations.

1.2 Prerequisites

In this section we collect several elementary facts about function spaces, which we basically consider to be known. This section is dynamic throughout the semester und will be updated according to our needs.

The necessary prerequisites which we assume to be known are the obligatory lectures Analysis I+II, in particular multivariable calculus and the most

[6] *Sobolev-spaces*
[7] $D^k u$ is the vector of all ordered k-th partial derivatives of u.

important existence theorems (inverse and implicit function theorem) and the most important elements of measure theory, such as the Lebesgue integral and L^p-spaces. Some of these things can be repeated during the exercises. As we have already done at some points, we will from time to time fix some notation and terminology from the Analysis courses in the footnotes, in case of possible ambiguities.

In the rest of this section we collect some important structures and spaces which will be used throughout this course.

General notation

(i) For the number systems we use the following notation:

- $\mathbb{R}$ the real numbers
- $\mathbb{C}$ the complex numbers
- $\mathbb{N}$ the positive integers
- $\mathbb{N}_0$ the non-negative integers.

(ii) The notion of a *multiindex* simplifies notation of partial derivatives a lot. A *multiindex* is simply an n-tuple of nonnegative integers

$$\alpha = (\alpha_1, \dots, \alpha_n).$$

Its *length* is

$$\langle\alpha\rangle = \sum_{i=1}^{n} \alpha_i.$$

For a vector $x \in \mathbb{R}^n$ we define

$$x^\alpha = \prod_{i=1}^{n} (x^i)^{\alpha_i}$$

and similarly the α-th partial derivative of an $\langle\alpha\rangle$-times continuously differentiable function u is defined by

$$\partial_\alpha u = \frac{\partial^{\langle\alpha\rangle} u}{\partial(x^1)^{\alpha_1} \dots \partial(x^n)^{\alpha_n}}.$$

Structures

1.2.1 Definition. (i) A pair (M, d) with a set M and a map

$$d \colon M \times M \to [0, \infty)$$

is called *metric space*, if for all $x, y, z \in M$ there hold:

(a) $d(x, y) = 0 \quad \Leftrightarrow \quad x = y$

(b) $d(x, y) = d(y, x)$

(c) $d(x, y) \leq d(x, z) + d(z, y)$

(ii) (M, d) is called *complete*, if every Cauchy sequence converges to a limit $x \in M$.

The concepts of convergence, Cauchy-sequence, completeness and boundedness in metric spaces should be clear according to your knowledge about $\mathbb{R}^n$ and will be assumed to be known. A good source to repeat these things are the lecture notes to Analysis by Prof. Kuwert, [9]. In the following $\mathbb{K}$ always stands for $\mathbb{R}$ or $\mathbb{C}$.

1.2.2 Definition. (i) A pair $(E, \|\cdot\|)$ with a $\mathbb{K}$-vector space E and a map

$$\|\cdot\|\colon E \to [0, \infty)$$

is called *normed vector space*, if for all $x, y \in E$ and all $\lambda \in \mathbb{K}$ there hold:

(a) $\|x\| = 0 \quad \Leftrightarrow \quad x = 0$

(b) $\|\lambda x\| = |\lambda|\|x\|$

(c) $\|x + y\| \leq \|x\| + \|y\|$

(ii) E is called *Banach space*, if it is complete as a metric space (with the induced metric $d(x, y) = \|x - y\|$).

1.2.3 Definition. (i) A pair (E, g) with a $\mathbb{K}$-vector space and a map

$$g\colon E \times E \to \mathbb{K}$$

is called *inner product space*, if for all $x, y, z \in E$ and all $\lambda, \mu \in \mathbb{K}$ there hold:

(a) $g(x, y) = \overline{g(y, x)}$

(b) $g(\lambda x + \mu y, z) = \lambda g(x, z) + \mu g(y, z)$

(c) $g(x, x) \geq 0$ and $(g(x, x) = 0 \quad \Leftrightarrow \quad x = 0)$

(ii) E is called *Hilbert space*, if E is a Banach space with respect to the induced norm

$$\|\cdot\|_g = \sqrt{g(\cdot, \cdot)}.$$

1.2.4 Notation. (i) The symbol $\langle \cdot, \cdot \rangle$ will always denote the Euclidean standard inner product of $\mathbb{K}^n$,

$$\langle x, y \rangle = \sum_{i=1}^{n} x^i \bar{y}^i.$$

(ii) We stipulate Ω to always represent an open set of $\mathbb{R}^n$, $n \geq 1$, equipped with the standard Euclidean inner product and the Lebesgue measure $\mathcal{L}^n$.

(iii) Open sets Ω, Ω' are defined to be related by the symbol

$$\Omega' \Subset \Omega,$$

if and only if the closure of Ω' is compact and contained in Ω.

(iv) An open ball of radius $r > 0$ around a point $x \in \mathbb{R}^n$ is denoted by

$$B_r(x) = \{y \in \mathbb{R}^n \colon |y - x| < r\}$$

and an open and connected subset of $\mathbb{R}^n$ is called *domain*.

(v) The symbol $|\cdot|$ always denoted the norm induced by $\langle\cdot,\cdot\rangle$ on $\mathbb{K}^n$,

$$|x| = \sqrt{\sum_{i=1}^{n}(x^i)^2}.$$

In both cases we do not fix the dimension n within the notation.

1.2.5 Notation (Einsteins summation convention)**.** We use Albert Einsteins *simplification (!!!)*, that in product expressions we sum over the same indices, if they *appear precisely once as superscript and subscript.* The range of summation is always the maximal possible range. For example take $x = (x^i), y = (y^i) \in \mathbb{R}^n$, then

$$\langle x, y\rangle = \sum_{i=1}^{n} x^i y^i = \delta_{ij} x^i y^j, \tag{1.2}$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

is called *Kronecker-delta.* Note that in the middle expression in (1.2) the summation sign is necessary, since both indices i are superscripts. The summation convention says, that in the expression on the right hand side of (1.2) we have to sum over i and j. Especially when using multilinear maps this leads to a notational simplification, e.g. in a term of the form

$$a_{i_1 \dots i_k} v^{i_1} \cdots v^{i_k},$$

where we have to sum over all indices $i_1, \dots, i_k$.

Function spaces

1.2.6 Definition. Let X be a set and V be a $\mathbb{K}$-vector space. The we define by V^X the vector space[8] of all maps

$$u \colon X \to V.$$

1.2.7 Remark (C^k-spaces). Let $n, m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be open and $k \in \mathbb{N}_0$.

(i) By $C^k(\Omega, \mathbb{R}^m)$ we denote the vector space of all k-times continuously differentiable functions

$$u \colon \Omega \to \mathbb{R}^m,$$

where for $k = 0$ we mean the space of continuous functions.

(ii) By $C^k(\bar{\Omega}, \mathbb{R}^m)$ we denote the vector space of all functions $u \in C^k(\Omega, \mathbb{R}^m)$ such that u and all its derivatives up to order k can be extended to $\bar{\Omega}$ continuously. We also write

$$C^k(\Omega) := C^k(\Omega, \mathbb{R}), \quad C^k(\bar{\Omega}) := C^k(\bar{\Omega}, \mathbb{R}).$$

[8] $(u + \lambda v)(x) := u(x) + \lambda v(x)$

If Ω is bounded and $k < \infty$, then $C^k(\bar{\Omega})$ equipped with the norm

$$|u|_{k,\Omega} := \sum_{i=0}^{k} \sup_{x\in\bar{\Omega}} |D^i u(x)|$$

is a Banach space.

1.2.8 Exercise. Prove that $C^k(\bar{\Omega})$ is a Banach space for all $k < \infty$ and bounded Ω.

1.2.9 Remark (L^p-spaces). Let $n, m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open and $1 \le p \le \infty$.

(i) By $L^p(\Omega, \mathbb{R}^m)$ we denote the vector space of equvialence classes of measurable functions $u\colon \Omega \to \mathbb{R}^n$ which are p-integrable in case $p < \infty$, i.e.

$$\|u\|_{p,\Omega} := \left(\int_\Omega |u|^p\right)^{\frac{1}{p}} < \infty,$$

or which are essentially bounded in case $p = \infty$, i.e.

$$\|u\|_{\infty,\Omega} = \inf\{c \ge 0\colon \mathcal{L}^n(\{x \in \Omega\colon |u(x)| > c\}) = 0\},$$

with the equivalence

$$u \sim v \quad \Leftrightarrow \quad \mathcal{L}^n(\{x \in \Omega\colon u(x) \neq v(x)\}) = 0.$$

The spaces $(L^p(\Omega, \mathbb{R}^m), \|\cdot\|_{p,\Omega})$ are Banach spaces for all $1 \le p \le \infty$. We also write

$$L^p(\Omega) := L^p(\Omega, \mathbb{R}).$$

If u is a function representing an element of an equivalence class in $L^p(\Omega)$, we will also use the symbol u to denote the whole equivalence class. This simplifies notation, but one should keep in mind, that the pointwise evaluation map at a point x,

$$u \mapsto u(x)$$

is not well defined.

(ii) The most important estimate in this context is *Hölder's inequality*, which states that for $1 \le p_i \le \infty$, $1 \le i \le k$, with

$$\sum_{i=1}^{k} \frac{1}{p_i} = 1$$

there holds

$$\forall u_i \in L^{p_i}(\Omega)\colon \int_\Omega \prod_{i=1}^{n} u_i \le \prod_{i=1}^{n} \|u_i\|_{p_i,\Omega}.$$

(iii) Let $0 < \mathcal{L}^n(\Omega) < \infty$. Then for every measurable function $u\colon \Omega \to \mathbb{R}$ there hold for all $1 \le p \le q \le \infty$

(a) $\|u\|_{p,\Omega} \le \mathcal{L}^n(\Omega)^{\frac{1}{p}-\frac{1}{q}} \|u\|_{q,\Omega}$.

(b) $\lim_{p\to\infty} \|u\|_{p,\Omega} = \|u\|_{\infty,\Omega}$.

(iv) We also define the local L^p spaces (which are not normed spaces).

$$L^p_{\text{loc}}(\Omega) = \{u \in \mathbb{R}^\Omega : u \in L^p(\Omega') \quad \forall \Omega' \Subset \Omega\}.$$

(v) For $p = 2$, $L^2(\Omega)$ is a Hilbert space with the inner product

$$\langle u, v\rangle_{2,\Omega} = \int_\Omega uv.$$

Different "levels" of continuity and differentiability are encoded in the following function spaces, which are called *Hölder spaces.* These play a crucial role in the solvability theory of PDE.

1.2.10 Remark ($C^{k,\alpha}$-spaces). Let $n, m, k \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open, $x_0 \in \Omega$, $0 < \alpha \leq 1$ and $u\colon \Omega \to \mathbb{R}^m$.

(i) u is called *locally in Ω Hölder continuous with exponent α*, if for every $\Omega' \Subset \Omega$

$$[u]_{\alpha,\Omega'} = \sup_{x,y\in\Omega', x\neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < \infty.$$

The space of such functions is denoted by $C^{0,\alpha}(\Omega, \mathbb{R}^m)$.

(ii) u is called *Hölder continuous in Ω with exponent α*, if

$$[u]_{\alpha,\Omega} = \sup_{x,y\in\Omega, x\neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < \infty.$$

The space of such functions is denoted by $C^{0,\alpha}(\bar{\Omega}, \mathbb{R}^m)$.

(iii) We also define

$$C^{k,\alpha}(\Omega, \mathbb{R}^m) := \{u \in C^k(\Omega, \mathbb{R}^m) : u_{,\beta} \in C^{0,\alpha}(\Omega, \mathbb{R}^m)\ \forall \langle\beta\rangle = k\}$$

and

$$C^{k,\alpha}(\bar{\Omega}, \mathbb{R}^m) := \{u \in C^k(\bar{\Omega}, \mathbb{R}^m) : u_{,\beta} \in C^{0,\alpha}(\bar{\Omega}, \mathbb{R}^m)\ \forall \langle\beta\rangle = k\}.$$

(iv) For $k \in \mathbb{N}_0$, on $C^{k,\alpha}(\bar{\Omega}, \mathbb{R}^m)$ we define the following norm:

$$|u|_{k,\alpha,\Omega} := |u|_{k,\Omega} + \max_{\langle\beta\rangle = k} [u_{,\beta}]_{\alpha,\Omega}.$$

(v) We write

$$C^{k,\alpha}(\Omega) = C^{k,\alpha}(\Omega, \mathbb{R}), \quad C^{k,\alpha}(\bar{\Omega}) = C^{k,\alpha}(\bar{\Omega}, \mathbb{R})$$

and also define

$$C^{k,0}(\Omega, \mathbb{R}^m) := C^k(\Omega, \mathbb{R}^m), \quad C^{k,0}(\bar{\Omega}, \mathbb{R}^m) := C^k(\bar{\Omega}, \mathbb{R}^m)$$

(vi) For $\alpha = 1$ the Hölder continuous functions with exponent α are also called Lipschitz continuous.

1.2.11 Exercise. Let $\Omega \Subset \mathbb{R}^n$. Prove that $C^{k,\alpha}(\bar{\Omega})$ is a Banach space for all $k \in \mathbb{N}_0$ and $0 < \alpha \leq 1$.

1.3 Domains in Euclidean space

Straightening the boundary

We have already seen, that the unique solvability of a PDE in a bounded domain $\Omega \subset \mathbb{R}^n$, e.g. of the Laplace equation

$$\Delta u = 0,$$

usually requires the prescription of boundary values, e.g. $u_{|\partial\Omega} = \varphi$. Then a central question in the theory of PDE is, how regular a solution u will be and if one can estimate its derivatives. These *a priori estimates* are then a useful tool in solving the existence problem. For example, if φ is smooth[9], we would expect

$$u \in C^\infty(\bar{\Omega}).$$

In order to prove this we need estimates of all derivatives up to the boundary $\partial\Omega$. Since such boundaries can have a very complicated curved structure, the *straightening of the boundary* is a useful tool. Basically this is a very special *coordinate system.*

1.3.1 Definition (Coordinate system)**.** Let $n \geq 1$, $\Omega \subset \mathbb{R}^n$ open and $k \geq 1$. A *coordinate system* of class C^k is a C^k-diffeomorphism[10]

$$\psi \colon \Omega \to \psi(\Omega) \subset \mathbb{R}^n.$$

For $x \in \Omega$ we call the components $\tilde{x}^i(x)$ of $\psi(x) = (\tilde{x}^i(x))_{1 \leq i \leq n}$ the $\tilde{x}$-*coordinates* of x.

1.3.2 Definition (C^k-boundary)**.** Let $n \geq 1$, $\Omega \subset \mathbb{R}^n$ a domain and $k \geq 1$.

(i) We say that, Ω has a C^k-*boundary*, $\partial\Omega \in C^k$, if for every $x_0 \in \partial\Omega$ there exists a ball $B_r(x_0)$ and a C^k-coordinate system

$$\psi \colon B_r(x_0) \to \psi(B_r(x_0)),$$

such that

$$\psi(x_0) = 0, \quad \psi(\partial\Omega \cap B_r(x_0)) = \{\tilde{x}^n = 0\} \cap \psi(B_r(x_0)),$$

$$\psi(\Omega \cap B_r(x_0)) = \{\tilde{x}^n > 0\} \cap \psi(B_r(x_0)).$$

(ii) The *tangent space* in $x_0 \in \partial\Omega$ is then defined by

$$T_{x_0}(\partial\Omega) = D\psi^{-1}(0)(\mathbb{R}^{n-1} \times \{0\}).$$

(iii) We define the *outer normal* $\nu(x_0)$ to $\partial\Omega$ in x_0 by the properties

$$\nu(x_0) \perp T_{x_0}(\partial\Omega), \quad |\nu(x_0)| = 1, \quad \exists \epsilon \; \forall 0 < t < \epsilon \colon x_0 + t\nu(x_0) \notin \bar{\Omega}.$$

[9] differentiable infinitely often

[10] ψ is invertible and $\psi, \psi^{-1} \in C^k$.

1.3.3 Remark. (i) From the definition of ψ und ν we obtain

$$\langle D\psi(x_0)\nu(x_0), e_n\rangle < 0.$$

By composing ψ with the linear map defined by

$$A(v) = \begin{cases} e_i, & v = e_i, \quad 1 \le i \le n-1 \\ -e_n, & v = D\psi(x_0)\nu(x_0) \end{cases}$$

we may suppose without loss of generality that

$$D\psi(x_0)\nu(x_0) = -e_n.$$

(ii) By restricting ψ to a smaller ball, we may also suppose that the C^k-norm of all component functions of ψ and ψ^{-1} are bounded.

1.3.4 Exercise. Prove that the tangent space $T_{x_0}(\partial\Omega)$ does not depend on the choice of the diffeomorphism ψ around x_0, which straightens the boundary.

The following example should be known from the analysis courses.

1.3.5 Exercise. Use polar coordinates to straighten the boundary of

$$\Omega = B_1(0) \subset \mathbb{R}^2$$

locally around a point (x_0, y_0) with $x_0 > 0$, i.e. find a suitable ψ. Calculate the tangent space $T_{(x_0,y_0)}(\partial B_1(0))$ and the corresponding outer normal.

In order to reduce global properties (such in whole of $\bar{\Omega}$) to local ones (such in domains with a straightened boundary), the *partition of unity* is a useful tool. Before we can prove this theorem, we need the following lemma.

1.3.6 Lemma. *Let $x_0 \in \mathbb{R}^n$ and $r > 0$. Then there exists a function*

$$0 \le \zeta \in C_c^\infty(B_{3r}(x_0))$$

satisfying

$$\zeta_{|\bar{B}_r(x_0)} = 1.$$

Proof. The function

$$f(t) = \begin{cases} e^{-\frac{1}{t}}, & t > 0 \\ 0, & t \le 0 \end{cases}$$

is smooth. The function

$$\zeta(x) = \frac{f\left(2 - \frac{|x-x_0|}{r}\right)}{f\left(2 - \frac{|x-x_0|}{r}\right) + f\left(\frac{|x-x_0|}{r} - 1\right)}$$

has the desired properties.

□

1.3.7 Theorem (Finite partition of unity for compact sets)**.** *Let $n \in \mathbb{N}$ and $K \subset \mathbb{R}^n$ be compact. Let V_j, $j \in \mathbb{N}$, be an open cover of K,*

$$K \subset \bigcup_{j=1}^{\infty} V_j.$$

Then there exist maps $\eta_i \in C_c^\infty(\mathbb{R}^n)$, $1 \le i \le m$, such that for all i there exists j with

$$\operatorname{supp} \eta_i \subset V_j,$$

$$\sum_{i=1}^{m} \eta_i(x) = 1 \quad \forall x \in K,$$

and

$$0 \le \eta_i \le 1.$$

Proof. Let $x \in K$, then $x \in V_j$ for some j. Since V_j is open, there exists a ball $B_{3r_x}(x)$ with

$$x \in B_{3r_x}(x) \subset V_j.$$

Hence

$$K \subset \bigcup_{x \in K} B_{r_x}(x)$$

and by compactness we can cover K with finitely many of these balls,

$$K \subset \bigcup_{i=1}^{m} B_{r_i}(x_i) =: U.$$

For every i set ζ_i to be the function from Lemma 1.3.6. For $x \in U$ define

$$\eta_i(x) = \frac{\zeta_i(x)}{\sum_{k=1}^{m} \zeta_k(x)}.$$

From $x \in U$ we deduce $x \in B_k$ for some k and hence $\zeta_k(x) = 1$. Thus the denominator is positive and η_i smooth in U. The three desired properties are obvious. □

Later we will need a version for open sets.

1.3.8 Theorem (Partition of unity for open sets)**.** *Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be open. Let $(U_j)_{j \in \mathbb{N}}$ be a covering of Ω such that*

$$\forall j \in \mathbb{N} \colon U_j \Subset \Omega.$$

Then there exist maps $\eta_i \in C_c^\infty(\mathbb{R}^n)$, $i \in \mathbb{N}$, such that for all i there exists j with

$$\operatorname{supp} \eta_i \subset U_j,$$

$$\sum_{i \in \mathbb{N}} \eta_i(x) = 1 \quad \forall x \in \Omega,$$

and

$$0 \le \eta_i \le 1.$$

Proof. First we have to modify the open cover (U_j) suitably. Set

$$V_{-1} = V_0 = \emptyset, \quad V_1 = U_1$$

and inductively suppose that V_k is already constructed for a given $k \in \mathbb{N}$. Pick an integer N_k, such that

$$\bar{V}_k \subset \bigcup_{j=1}^{N_k} U_j =: V_{k+1}.$$

Furthermore we may take $N_{k+1} > N_k$ in each step. Hence we have produced an open covering $(V_k)_{k\in\mathbb{N}}$ of Ω with

$$\forall k \in \mathbb{N} \colon \bar{V}_k \subset V_{k+1}.$$

However, we still need to ensure that every point $x \in \Omega$ lies in only finitely many of the covering open sets. Thus we define for $k \in \mathbb{N}_0$

$$W_k = V_{k+2} \backslash \bar{V}_k.$$

For $x \in \Omega$ there exists a minimal $k \in \mathbb{N}$ with $x \in \bar{V}_k \subset V_{k+1}$ and hence

$$x \in W_{k-1}.$$

Thus

$$\Omega = \bigcup_{k=0}^{\infty} W_k$$

and every closed ball $\bar{B}_r(x) \subset \Omega$ intersects only finitely many of the W_k. Now we construct the partition of unity. Let

$$x \in \overline{W}_k \subset V_{k+3} \backslash \bar{V}_{k-1},$$

then $x \in U_j$ for some j and there exists r_x such that

$$B_{3r_x}(x) \subset U_j \cap V_{k+3} \backslash \bar{V}_{k-1}. \tag{1.3}$$

Thus for finitely many $x_l \in \overline{W}_k$ there holds

$$\overline{W}_k \subset \bigcup_{l=1}^{L_k} B_{r_l}(x_l).$$

Collecting these balls for each k, we get a countable collection of balls $B_{r_i}(x_i)$ with $\bar{B}_{3r_i}(x_i) \Subset \Omega$. For every i set ζ_i to be the function from Lemma 1.3.6 and for $x \in \Omega$ define

$$\eta_i(x) = \frac{\zeta_i(x)}{\sum_{k=1}^{\infty} \zeta_k(x)}.$$

This is well-defined, since in the denominator the sum is always finite, due to (1.3). The proof is complete. □

The surface measure

1.3.9 Definition. Let $n \geq 1$, $\Omega \subset \mathbb{R}^n$ open and

$$\psi = (\tilde{x}^i) \colon \Omega \to V := \psi(\Omega)$$

be a C^1-coordinate system. On V we define the *Gramian matrix associated to* ψ, $g = (g_{ij})_{1 \leq i,j \leq n}$, by

$$g(\tilde{x}) = (D\psi^{-1}(\tilde{x}))^T D\psi^{-1}(\tilde{x}) \tag{1.4}$$

with the components

$$g_{ij}(\tilde{x}) = \left\langle \frac{\partial \psi^{-1}}{\partial \tilde{x}^i}(\tilde{x}), \frac{\partial \psi^{-1}}{\partial \tilde{x}^j}(\tilde{x}) \right\rangle.$$

With the help of the Gramian matrix we can define the surface measure on $\partial\Omega$. However we first have to define which sets are measurable.

1.3.10 Definition. Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ open with C^1-boundary. We call $E \subset \partial\Omega$ measurable, if

$$\mathrm{pr}_{\mathbb{R}^{n-1}} \left(\psi(E \cap B_r(x_0))\right) \subset \mathrm{pr}_{\mathbb{R}^{n-1}} \left(\mathbb{R}^{n-1} \times \{0\}\right)$$

is measurable with respect to the $(n-1)$-dimensional Lebesgue measure for every local straightening function ψ around $x_0 \in \partial\Omega$. Here

$$\begin{aligned} \mathrm{pr}_{\mathbb{R}^n} \colon \mathbb{R}^n &\to \mathbb{R}^{n-1} \\ (x^1, \ldots, x^n) &\mapsto (x^1, \ldots, x^{n-1}). \end{aligned}$$

1.3.11 Remark. (i) Since ψ is bijective, ψ interchanges with all set operations and hence the set of measurable sets $\mathcal{A}$ forms a σ-algebra on $\partial\Omega$.

(ii) Let $E \subset \partial\Omega$ be open[11], then $E \in \mathcal{A}$, since for all ψ we have

$$\psi(E \cap B_r(x_0)) = \psi(U \cap \partial\Omega \cap B_r(x_0)) = \psi(U \cap B_r(x_0)) \cap \mathbb{R}^{n-1} \times \{0\}.$$

The projection of the latter set is open in $\mathbb{R}^{n-1}$ and hence Lebesgue measurable.

1.3.12 Theorem (Surface measure)**.** *Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ a bounded domain with C^1-boundary. There exists a uniquely determined measure μ on $\partial\Omega$, such that for all $\mathcal{A}$-measurable sets $E \subset \partial\Omega$, that lie in the domain of definition $B_r(x_0)$ of a local straightening function ψ, there holds*

$$\mu(E) = \int_{\psi(E)} \sqrt{\det g_{\partial\Omega}}\, d\mathcal{L}^{n-1}, \tag{1.5}$$

where $g_{\partial\Omega}$ is the Gramian matrix associated to $\psi_{|\partial\Omega}$ [12]*. We call μ the surface measure on $\partial\Omega$.*

[11] i.e. there exists an open set $U \subset \mathbb{R}^n$ with $E = U \cap \partial\Omega$

[12] This one is defined by the formula (1.4), where we replace ψ by $\psi_{|\partial\Omega}$

Proof. (i) Let $E \in \mathcal{A}$, $E \subset B_r(x_0)$ and $\psi \in C^1(B_r(x_0), \psi(B_r(x_0)))$ be a local straightening function. We define $\mu(E)$ by the formula (1.5). Then $\mu(E)$ is well defined, since $\psi(E)$ is Lebesgue measurable, $\sqrt{\det g_{\partial\Omega}}$ is bounded and for another straightening function

$$(\hat{x}^i) = \phi \colon B_R(y_0) \to \phi(B_R(y_0))$$

with Gramian matrix $h = h_{\partial\Omega}$ associated to $\phi_{|\partial\Omega}$ there hold (for better readability we omit the arguments here):

$$\begin{aligned} h = (D\phi^{-1})^T D\phi^{-1} &= (D\psi^{-1} \circ D(\psi \circ \phi^{-1}))^T (D\psi^{-1} \circ D(\psi \circ \phi^{-1})) \\ &= D(\psi \circ \phi^{-1})^T \circ (D\psi^{-1})^T \circ D\psi^{-1} \circ D(\psi \circ \phi^{-1}), \end{aligned}$$

$$\det h(\hat{x}) = \det(D(\psi \circ \phi^{-1})(\hat{x}))^2 \det g(\psi \circ \phi^{-1}(\hat{x})),$$

where $g = g_{\partial\Omega}$, and hence

$$\int_{\phi(E)} \sqrt{\det h} = \int_{\phi(E)} \sqrt{\det g}\ |\det(D(\psi \circ \phi^{-1}))| = \int_{\psi(E)} \sqrt{\det g}$$

due to the transformation theorem.

(ii) Since $\partial\Omega$ is compact, there exist finitely many local straightening functions

$$\psi_i \colon B_i := B_{r_i}(x_i) \to \psi_i(B_i), \quad 1 \le i \le N,$$

with

$$\partial\Omega \subset \bigcup_{i=1}^N B_i.$$

Put

$$W_1 = B_1 \cap \partial\Omega, \quad W_i = (B_i \cap \partial\Omega) \backslash \bigcup_{k=1}^{i-1} W_k.$$

The $W_i \subset B_i$ are disjoint, measurable and cover $\partial\Omega$. Let $E \in \mathcal{A}$ be arbitrary, then

$$E = \bigcup_{i=1}^N (E \cap W_i).$$

Hence, if a measure μ with the desired property exists, it can only have the form

$$\mu(E) = \sum_{i=1}^N \mu(E \cap W_i) = \sum_{i=1}^N \int_{\psi_i(E \cap W_i)} \sqrt{\det g^i},$$

which proves the uniqueness.

(iii) We see immediately, that by this formula a measure is defined and that it has the desired property due to the calculation in (i). □

1.3.13 Corollary (Surface integral)**.** *Let $n \ge 1$ and Ω a bounded domain with C^1-boundary. Let ψ be a local straightening function on $B = B_r(x_0)$ and*

$$f \colon \partial\Omega \to \mathbb{R}$$

$\mathcal{A}$-measurable[13] *and $f(x) = 0$ for all $x \notin \partial\Omega \cap B$. Then f is μ-integrable*[14] *if and only if $f \circ \psi^{-1}\sqrt{\det g_{\partial\Omega}}$ is integrable with respect to the $(n-1)$-dimensional Lebesgue-measure and there holds*

$$\int_{\partial\Omega} f \; d\mu = \int_{\psi(\partial\Omega\cap B)} f \circ \psi^{-1}\sqrt{\det g_{\partial\Omega}} \; d\mathcal{L}^{n-1}.$$

Proof. Let $E \in \mathcal{A}$ with $E \subset B$ and for the moment $f = \chi_E$[15]. Due to (1.5) there holds

$$\int_{\partial\Omega} \chi_E = \mu(E) = \int_{\psi(E)} \sqrt{\det g_{\partial\Omega}} = \int_{\psi(\partial\Omega\cap B)} \chi_E \circ \psi^{-1}\sqrt{\det g_{\partial\Omega}}.$$

Hence the result holds for characteristic functions. Due to the linearty of the integral it also holds for all step functions. With the theorem of monotone convergence (Beppo-Levi) the result holds for all nonnegative functions and by decomposition into positive and negative part for all measurable functions. ☐

1.4 The Gaussian divergence theorem

In this section we prove the Gaussian divergence theorem. We will only do this for domains with C^2-boundary. With a little more effort one can also prove it for C^1-boundaries and with even more effort one can prove it for Lipschitz boundaries. First we need definition.

1.4.1 Definition (Vector field)**.** Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be open.

(i) A map

$$X \in C^k(\Omega, \mathbb{R}^n)$$

is called vector field of class C^k.

(ii) Let ψ be a C^1-coordinate system and X a C^1-vector field on Ω. Then

$$\tilde{X}(\psi(x)) = D\psi(x)X(x)$$

is called the *image vector field* of X under $D\psi$.

1.4.2 Definition (Divergence)**.** Let $n \geq 1$, $\Omega \subset \mathbb{R}^n$ be open and $X \in C^1(\Omega, \mathbb{R}^n)$ a vector field. We define the *divergence* of X by

$$\operatorname{div} X = \frac{\partial X^i}{\partial x^i}.$$

1.4.3 Lemma. *(i) Let $\epsilon > 0$ and let $g = g(t)$, $t \in (-\epsilon, \epsilon)$, be a one-parameter family of invertible $(n \times n)$-matrices, which is differentiable with respect to t. Then*

$$(\partial_t \det g)_{|t=0} = \det g(0) \operatorname{tr}\left(g^{-1}(0) \circ \dot{g}(0)\right) \tag{1.6}$$

[13] I.e. $f^{-1}(V) \in \mathcal{A}$ for all open $V \subset \mathbb{R}$

[14] $\int_{\partial\Omega} |f| \; d\mu < \infty$

[15]
$$\chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

(ii) In an arbitrary C^2-coordinate system $\psi = (\tilde{x}^i)$ the divergence is given by

$$(\operatorname{div} X) \circ \psi^{-1} = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial \tilde{x}^i} \left(\sqrt{\det g}\ \tilde{X}^i \right), \tag{1.7}$$

where g is the Gramian matrix associated to ψ and $\tilde{X}$ is the image vector field of X under $D\psi$.

Proof. (i) First suppose that $g(0) = \mathrm{id}$. Write

$$\det g = \sum_\pi \prod_{i=1}^n (\operatorname{sgn} \pi) g_{i\pi(i)}.$$

Then

$$\partial_t \det g = \sum_\pi (\operatorname{sgn} \pi) \sum_{k=1}^n g_{1\pi(1)} \cdots \dot{g}_{i\pi(i)} \cdots g_{n\pi(n)}.$$

Since $g(0) = \mathrm{id}$, such a product term is only nonzero if π is the identity permutation. Hence

$$(\partial_t \det g)_{|t=0} = \operatorname{tr} \dot{g}(0)$$

and the results holds in this special case. In the general case apply the special case to the function

$$h(t) = g(0)^{-1} g(t).$$

(ii) We also write

$$\psi^{-1}(\tilde{x}) = (x^i(\tilde{x})).$$

The components of $\tilde{X}$ are given by

$$\tilde{X}^i = \frac{\partial \tilde{x}^i}{\partial x^k}(\psi^{-1}(\tilde{x})) X^k(\psi^{-1}(\tilde{x})).$$

Hence

$$\begin{aligned}\frac{\partial \tilde{X}^i}{\partial \tilde{x}^i} &= \frac{\partial^2 \tilde{x}^i}{\partial x^k \partial x^l} \frac{\partial x^l}{\partial \tilde{x}^i} X^k + \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial X^k}{\partial x^l} \frac{\partial x^l}{\partial \tilde{x}^i} \\ &= \frac{\partial^2 \tilde{x}^i}{\partial x^k \partial x^l} \frac{\partial x^l}{\partial \tilde{x}^i} X^k + \operatorname{div} X \circ \psi^{-1}.\end{aligned}$$

From (i) we obtain

$$\begin{aligned}\frac{1}{\sqrt{\det g}} \frac{\partial}{\partial \tilde{x}^i} \left(\sqrt{\det g}\ \tilde{X}^i \right) &= \frac{1}{2} g^{pq} \frac{\partial}{\partial \tilde{x}^i} g_{pq} \tilde{X}^i + \frac{\partial^2 \tilde{x}^i}{\partial x^k \partial x^l} \frac{\partial x^l}{\partial \tilde{x}^i} X^k + \operatorname{div} X \circ \psi^{-1} \\ &= g^{mq} \delta_{ls} \frac{\partial^2 x^l}{\partial \tilde{x}^m \partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^q} \tilde{X}^i - \frac{\partial^2 x^l}{\partial \tilde{x}^i \partial \tilde{x}^m} \frac{\partial \tilde{x}^m}{\partial x^l} \tilde{X}^i \\ &\quad + \operatorname{div} X \circ \psi^{-1} \\ &= \operatorname{div} X \circ \psi^{-1},\end{aligned}$$

since

$$g^{mq} \delta_{ls} \frac{\partial x^s}{\partial \tilde{x}^q} = \frac{\partial \tilde{x}^m}{\partial x^l},$$

which can be seen by testing these linear maps on all basis vectors $\partial x^l / \partial \tilde{x}^j$. □

The Gaussian divergence theorem is a generalization of the fundamental theorem of calculus and reads as follows.

1.4.4 Theorem (Gaussian divergence theorem). *Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ a bounded domain with C^2-boundary and outer normal ν. Let $X \in C^1(\bar{\Omega}, \mathbb{R}^n)$ be a vector field. Then there holds*

$$\int_\Omega \operatorname{div} X = \int_{\partial\Omega} \langle X, \nu \rangle .$$

Proof. First assume $\operatorname{supp} X$ to be contained in the domain of definition B of a straightening function ψ. Then

$$\begin{aligned}
\int_\Omega \operatorname{div} X &= \int_{\psi(\Omega\cap B)} \operatorname{div} X(\psi^{-1}(\tilde{x}))\sqrt{\det g(\tilde{x})}\; d\tilde{x} \\
&= \int_{\{\tilde{x}^n>0\}} \frac{\partial}{\partial \tilde{x}^j}\left(\sqrt{\det g(\tilde{x})}\tilde{X}^j(\tilde{x})\right)\; d\tilde{x} \\
&= \int_{\{\tilde{x}^n>0\}} \frac{\partial}{\partial \tilde{x}^n}\left(\sqrt{\det g(\tilde{x})}\tilde{X}^n\right)\; d\tilde{x} \\
&= -\int_{\mathbb{R}^{n-1}} \tilde{X}^n(\tilde{x}^1,\dots,\tilde{x}^{n-1},0)\sqrt{\det g(\tilde{x}^1,\dots,\tilde{x}^{n-1},0)}\; d\tilde{x}^1\cdots d\tilde{x}^{n-1} \\
&= \int_{\mathbb{R}^{n-1}} g(-e_n, \tilde{X})\sqrt{\det g_{\partial\Omega}} \\
&= \int_{\partial\Omega} \langle X, \nu\rangle .
\end{aligned}$$

We used here that

$$g_{in} = \left\langle \frac{\partial \psi^{-1}}{\partial \tilde{x}^i}, \frac{\partial \psi^{-1}}{\partial \tilde{x}^n} \right\rangle = \delta_{in}$$

and hence

$$\det g = \det g_{\partial\Omega}, \quad \tilde{X}^n = g(e_n, \tilde{X}).$$

Now let X be arbitrary and $(B_i)_{1\leq i\leq N} = (B_{r_i}(x_i))_{1\leq i\leq N}$ a family of balls that cover $\partial\Omega$ and such that $B_{3r_i}(x_i)$ are the domains of straightening functions. Also cover

$$\bar{\Omega} \backslash \bigcup_{i=1}^N B_i \subset \bigcup_{j=N+1}^K B_j$$

by balls $(B_i)_{N+1\leq i\leq K}$, such that

$$\bar{B}_i \cap \partial\Omega = \emptyset.$$

Let η_i be an associated partition of unity according to Theorem 1.3.7. Then

$$\int_\Omega \operatorname{div} X = \sum_{i=1}^K \int_\Omega \operatorname{div}(\eta_i X) = \sum_{i=1}^N \int_\Omega \operatorname{div}(\eta_i X) = \sum_{i=1}^N \int_{\partial\Omega} \langle \eta_i X, \nu\rangle = \int_{\partial\Omega} \langle X, \nu\rangle .$$

We also used, that for a vector field $Y \in C^1_c(\Omega)$ we always have

$$\int_\Omega \operatorname{div} Y = 0,$$

which follows from Fubini's theorem and the fundamental theorem of calculus.

□

From the divergence theorem we will deduce some more useful formulas.

1.4.5 Definition (Gradient and Laplace)**.** Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ open.

(i) If $u \in C^1(\Omega)$, then we define the *gradient of* u *in* $x \in \Omega$, $\nabla u(x) \in \mathbb{R}^n$, by the property

$$Du(x)X = \langle \nabla u(x), X \rangle \quad \forall X \in \mathbb{R}^n.$$

∇u has the components

$$\nabla^i u = \delta^{ij} \frac{\partial u}{\partial x^j}.$$

(ii) We define the *Laplace-operator* by

$$\begin{aligned} \Delta \colon C^2(\Omega) &\to C^0(\Omega) \\ u &\mapsto \Delta u = \operatorname{div}(\nabla u). \end{aligned}$$

We obtain immediately:

1.4.6 Exercise. Let $n \geq 1$, $\Omega \subset \mathbb{R}^n$ a bounded domain with C^2-boundary and outer normal ν. Let $u \in C^1(\bar{\Omega})$ and $v \in C^1(\bar{\Omega})$. Then

(i) (Partial integration)

$$\int_\Omega u \partial_{x^i} v = -\int_\Omega \partial_{x^i} u v + \int_{\partial\Omega} u v \nu^i.$$

If $u \in C^2(\bar{\Omega})$, additionally there hold

(ii) (1. Green's formula)

$$\int_\Omega \Delta u = \int_{\partial\Omega} \langle \nabla u, \nu \rangle$$

(iii) (2. Green's formula)

$$\int_\Omega v \Delta u = -\int_\Omega \langle \nabla v, \nabla u \rangle + \int_{\partial\Omega} v \langle \nabla u, \nu \rangle.$$

We want to represent Δ in a different coordinate system. Therefore we need a representation of the image vector field of the gradient under a coordinate transformation:

1.4.7 Lemma. *Let $n \geq 1$, $\Omega \subset \mathbb{R}^n$ open and $\psi = (\tilde{x}^i)$ a C^1-coordinate system. Then there holds*

$$\frac{\partial \tilde{x}^i}{\partial x^k} \nabla^k u(x) = g^{im}(\tilde{x}) \frac{\partial (u \circ \psi^{-1})}{\partial \tilde{x}^m} =: \tilde{\nabla}^i u,$$

where (g^{ij}) is the inverse of the Gramian matrix (g_{ij}) associated to ψ.

Proof. Let $X \in \mathbb{R}^n$ and $\tilde{X}$ be its image vector under $D\psi$. We calculate

$$\begin{aligned}
\left\langle \frac{\partial \psi^{-1}}{\partial \tilde{x}^i} g^{im} \frac{\partial (u \circ \psi^{-1})}{\partial \tilde{x}^m}, X \right\rangle &= \left\langle \frac{\partial \psi^{-1}}{\partial \tilde{x}^i} g^{im} \frac{\partial (u \circ \psi^{-1})}{\partial \tilde{x}^m}, \frac{\partial \psi^{-1}}{\partial \tilde{x}^j} \tilde{X}^j \right\rangle \\
&= g_{ij} g^{im} \frac{\partial (u \circ \psi^{-1})}{\partial \tilde{x}^m} \tilde{X}^j \\
&= \frac{\partial (u \circ \psi^{-1})}{\partial \tilde{x}^j} \tilde{X}^j \\
&= \frac{\partial u}{\partial x^k} \frac{\partial x^k}{\partial \tilde{x}^j} \frac{\partial \tilde{x}^j}{\partial x^m} X^m \\
&= \frac{\partial u}{\partial x^k} X^k \\
&= \langle \nabla u(x), X \rangle .
\end{aligned}$$

This holds for all $X \in \mathbb{R}^n$ and hence the claim follows. □

1.4.8 Exercise. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0\}$. Write the Laplace-operator in polar coordinates

$$(r, \theta) = \psi \colon \Omega \to \tilde{\Omega}.$$

This means that you have to find a differential operator L in $\tilde{\Omega}$ (which involves $\partial/\partial r$ and $\partial/\partial \theta$), such that

$$(\Delta u) \circ \psi^{-1} = L(u \circ \psi^{-1}).$$

Hint: Use (1.7).

Chapter 2

The maximum principle

The maximum principle is certainly one of the most important tools in the theory of partial differential equations and in whole analysis. In its very simplest form it appears in the scope of a well known fact:

A real function $f\colon (a,b) \to \mathbb{R}$ which satisfies $f'' > 0$ does not attain any local maximum.

The inequality $f'' > 0$ is a very simple differential inequality. This chapter is devoted to investigate to some extent, if such a result also holds for more complicated differential inequalities for functions of several variables, which also include certain combinations of partial derivatives. Finally such a result will hold for certain classes of PDE.

2.1 Linear elliptic and parabolic operators

In this section we define the class of equations, for which we will prove a maximum principle. These results will only be prototypes in the sense that in the literature there are more general versions, but to get a first impression the equations treated here will be general enough. The first type of equations are the so-called *elliptic* ones. We have already seen the Laplace-operator as our first elliptic operator. As already mentioned, we restrict to operators of second order acting on real valued functions u.

2.1.1 Definition. Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ open.

(i) A linear map L of the form

$$L\colon C^2(\Omega) \to \mathbb{R}^\Omega$$
$$Lu = a^{ij}u_{,ij} + b^i u_{,i} + du, \text{[1]}$$

where $(a^{ij}) \in (\mathbb{R}^{n^2})^\Omega$ is symmetric, $(b^i) \in (\mathbb{R}^n)^\Omega$ and $d \in \mathbb{R}^\Omega$, is called *elliptic in* $x \in \Omega$, if $a^{ij}(x)$ is positive definite, i.e.

$$\exists \lambda(x) > 0 \ \forall (\xi_i) \in \mathbb{R}^n \colon a^{ij}(x)\xi_i\xi_j \geq \lambda(x)|\xi|^2.$$

[1] We try to keep notation as slim as possible, with as few symbols as possible. Hence from now on we use the convention, that *indices appearing after a comma denote partial*

L is called *linear elliptic operator in* Ω, if L is elliptic at every $x \in \Omega$.

(ii) Let $S \subset \Omega$. L is called *strictly elliptic in* S, if

$$\exists \lambda > 0\ \forall x \in S\ \forall (\xi_i) \in \mathbb{R}^n \colon a^{ij}(x)\xi_i\xi_j \geq \lambda |\xi|^2 \tag{2.1}$$

and *uniformly elliptic in* S, if

$$\exists \Lambda > \lambda > 0\ \forall x \in S\ \forall (\xi_i) \in \mathbb{R}^n \colon \lambda |\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda |\xi|^2. \tag{2.2}$$

The major, classical source for the theory of elliptic partial differential equations of second order is the excellent book by David Gilbarg and Neil Trudinger, [5]. Most of the proofs in this chapter are more or less taken from this book.

In the following we calculate how a linear partial differential operator of second order transforms under a change of coordinates, also compare Exercise 1.4.8.

2.1.2 Proposition. *Let* $n \geq 1$, $\Omega \subset \mathbb{R}^n$ *open,* L *a differential operator of the form*

$$Lu = a^{ij}u_{,ij} + b^i u_{,i} + du$$

and

$$\psi = (\tilde{x}^i) \colon \Omega \to \tilde{\Omega}$$

a C^2*-coordinate transformation, then there holds for all* $u \in C^2(\Omega)$*, that*

$$\tilde{L}(u \circ \psi^{-1}) = (Lu) \circ \psi^{-1},$$

with a differential operator

$$\tilde{L} = \tilde{a}^{ij}\tilde{\partial}_{ij} + \tilde{b}^i\tilde{\partial}_i + \tilde{d},$$

where

$$\tilde{a}^{ij} = \left(a^{kl}\tilde{x}^i_{,k}\tilde{x}^j_{,l}\right) \circ \tilde{x}^{-1},$$

$$\tilde{b}^i = \left(b^k\tilde{x}^i_{,k} + a^{kl}\tilde{x}^i_{,kl}\right) \circ \tilde{x}^{-1}$$

and

$$\tilde{d} = d \circ \tilde{x}^{-1}.$$

Proof. Let $\tilde{u} \colon \tilde{\Omega} \to \mathbb{R}$ be defined by $\tilde{u}(\tilde{x}) = u \circ \psi^{-1}$. Hence $u(x) = \tilde{u}(\tilde{x}(x))$. We calculate

$$u_{,i} = \tilde{u}_{,k}\tilde{x}^k_{,i},$$

$$u_{,ij} = \tilde{u}_{,kl}\tilde{x}^k_{,i}\tilde{x}^l_{,j} + \tilde{u}_{,k}\tilde{x}^k_{,ij}$$

and hence

$$\begin{aligned} Lu &= a^{ij}u_{,ij} + b^i u_{,i} + du \\ &= a^{ij}\tilde{u}_{,kl}\tilde{x}^k_{,i}\tilde{x}^l_{,j} + a^{ij}\tilde{u}_{,k}\tilde{x}^k_{,ij} + b^i\tilde{u}_{,k}\tilde{x}^k_{,i} + du. \end{aligned}$$

The result follows. ∎

---

derivatives, e.g.

$$u_{,i} = \frac{\partial u}{\partial x^i}, \quad u_{,ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}, \quad \alpha_{i,j} = \frac{\partial \alpha_i}{\partial x^j}$$

etc. This is also well suitable for use of the summation convention.

Hence under a coordinate transformation such a differential operator is transformed to one of the same kind. If the derivatives of the coordinate transformation are under control, even the types (strictly, uniformly) elliptic carry over, as you can convince yourself in the next exercise.

2.1.3 Exercise. Suppose the C^2-coordinate transformation

$$\psi = (\tilde{x}^i) \colon \Omega \to \tilde{\Omega}$$

and its inverse ψ^{-1} both have bounded derivatives up to second order in a subset $S \subset \Omega$, then L is (strictly) [uniformly] elliptic in S if and only if $\tilde{L}$ is (strictly) [uniformly] elliptic in $\tilde{S} = \psi(S)$.

Now we define the parabolic operators, a special case of which already appeared in the heat equation. It has the special property that it contains a certain differential of first order in one direction. We distinguish this direction by splitting the domain and considering it to be a cartesian product $(0, T) \times \Omega$, where $\Omega \subset \mathbb{R}^n$ is open.

2.1.4 Definition. Let $n \geq 1$, $\Omega \subset \mathbb{R}^n$ be open, $T > 0$ a real number and $Q = (0, T) \times \Omega$.

(i) A linear map P of the form

$$P \colon C^{1;2}(Q) \to \mathbb{R}^Q$$
$$Pu = a^{ij} u_{,ij} + b^i u_{,i} + du - \dot{u},$$ [2]

where $(a^{ij}) \in (\mathbb{R}^{n^2})^Q$ is symmetric, $(b^i) \in (\mathbb{R}^n)^Q$ and $d \in \mathbb{R}^Q$, is called *parabolic in* $(t, x) \in Q$, if $a^{ij}(t, x)$ is positive definite. P is called *linear parabolic operator in* Q, if P is parabolic at every $(t, x) \in Q$.

(ii) Let $S \subset Q$. P is called *strictly (uniformly) parabolic in* S, if the relation eq. (2.1) (eq. (2.2)) holds with x replaced by (t, x).

Parabolic equations are often used to model real world phenomena, such as the flow of heat in a material, as we have already seen. Hence this is a very important class of equations. Standard textbooks which cover some theory of these equations are [2, 11].

2.2 Maximum principles

Weak maximum principles

In this section we prove a[3] maximum principle for linear elliptic and parabolic operators. We start with the weak maximum principle, which roughly states that solutions to certain PDE will attain their global maximum on the boundary of the given domain.

First we need a definition.

[2] $C^{1;2}(Q)$ is the space of functions which are once continuously differentiable with respect to t and twice with respect to x. Then we write $\dot{u} = \partial_t u$.

[3] We use 'a' and not 'the', because we will not prove it in the most general possible form.

2.2.1 Definition. Let $n \geq 1$, $Q \subset \mathbb{R}^{n+1}$ a set of the form $Q = (0,T) \times \Omega$ with $T > 0$ and $\Omega \subset \mathbb{R}^n$ open. The *parabolic boundary* $\partial_p Q$ of Q is defined by

$$\partial_p Q = \left(\{0\} \times \bar{\Omega}\right) \cup \left([0,T] \times \partial\Omega\right).$$

2.2.2 Theorem (Parabolic weak maximum principle)**.** *Let $n \geq 1$, $\Omega \subset \mathbb{R}^n$ open and bounded, $T > 0$ a real number and $Q = (0,T) \times \Omega$. Let*

$$Pu = a^{ij} u_{,ij} + b^i u_{,i} + du - \dot{u}$$

be a linear parabolic operator in Q with $d \leq 0$. Suppose $u \in C^{1;2}(Q) \cap C^0(\bar{Q})$ solves the inequality

$$Pu \geq 0.$$

Then

$$\max_{\bar{Q}} u \leq \max \left(0, \max_{\partial_p Q} u\right).$$

2.2.3 Remark. The statement of the weak maximum principle can be rephrased by saying that under the given assumptions, if u attains a positive maximum, this maximum is attained on the parabolic boundary.

Proof of Theorem 2.2.2. This proof is taken from [8]. For $(t,x) \in [0,T) \times \bar{\Omega}$ define

$$v(t,x) = u(t,x) - \frac{\epsilon}{T-t},$$

where $\epsilon > 0$. v satisfies in Q:

$$Pv = Pu - \frac{\epsilon d}{T-t} + \frac{\epsilon}{(T-t)^2}. \tag{2.3}$$

We first show that v attains positive maxima on the parabolic boundary. If there was a point

$$(t_0, x_0) \in \bar{Q} \backslash \partial_p Q$$

with

$$v(t_0, x_0) = \max_{\bar{Q}} v,$$

then first of all $t_0 < T$, hence $(t_0, x_0) \in Q$ and we may conclude

$$\dot{v}(t_0, x_0) = v_{,i}(t_0, x_0) = 0$$

and $D_x^2 v(t_0, x_0)$ is non-positive definit. Since $(a^{ij}(t_0, x_0))$ is positive definite, we have

$$a^{ij}(t_0, x_0) v_{,ij}(t_0, x_0) \leq 0.$$ [4]

Hence

$$Pv(t_0, x_0) \leq dv(t_0, x_0) \leq 0.$$

However we have by (2.3):

$$Pv(t_0, x_0) > 0,$$

[4] This is an exercise in linear algebra, which is recommended to be worked out.

a contradiction. Thus for all $(t,x) \in [0,T) \times \bar{\Omega}$ and all $\epsilon > 0$ we have

$$u(t,x) - \frac{\epsilon}{T-t} \leq \max\left(0, \max_{\partial_p Q \setminus \{t=T\}} v\right) \leq \max\left(0, \max_{\partial_p Q} u\right).$$

Letting $\epsilon \to 0$ gives the result. □

2.2.4 Remark. The case where d does not have a sign will be discussed in the exercises.

From the weak maximum principle we immediately obtain a uniqueness result for solutions of parabolic equations.

2.2.5 Corollary. *Under the assumptions of Theorem 2.2.2 suppose that* $u, w \in C^{1;2}(Q) \cap C^0(\bar{Q})$ *satisfy*

$$\begin{aligned} Pu &= Pw \text{ in } Q \\ u &= w \text{ on } \partial_p Q. \end{aligned}$$

Then $u = w$.

Proof. Apply Theorem 2.2.2 to $\pm(u-w)$ to obtain

$$\max_{\bar{Q}} |u - w| \leq 0.$$

A maximum principle for elliptic equations is also valid. We follow [5].

2.2.6 Theorem (Elliptic weak maximum principle). *Let* $n \geq 1$, $\Omega \subset \mathbb{R}^n$ *be open and bounded and*

$$Lu = a^{ij}u_{,ij} + b^i u_{,i} + du$$

be a linear elliptic operator in Ω *with* $d \leq 0$,

$$\forall x \in \Omega \; \exists \lambda(x) > 0 \; \forall (\xi_i) \in \mathbb{R}^n \colon a^{ij}(x)\xi_i\xi_j \geq \lambda(x)|\xi|^2$$

and bounded $\lambda^{-1}|b|$. *Suppose* $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *solves the inequality*

$$Lu \geq 0.$$

Then

$$\max_{\bar{\Omega}} u \leq \max\left(0, \max_{\partial\Omega} u\right).$$

Proof. Define for $\gamma > 0$

$$z(x) = e^{\gamma x^1}.$$

Then

$$(L-d)z = \gamma^2 a^{11} z + \gamma b^1 z > 0$$

for sufficiently large γ. Defne for $\epsilon > 0$

$$v(x) = u(x) + \epsilon z(x).$$

Then

$$(L-d)v = (L-d)(u + \epsilon z) > -du \geq 0$$

on the set $\Omega' = \{x \in \Omega\colon u(x) > 0\}$. Thus v does not attain positive local maxima in Ω', since at such points

$$(L-d)v = a^{ij}v_{,ij} + b^i v_{,i} \le 0.$$

Thus

$$\sup_{\bar\Omega'} v \le \max\left(0, \max_{\partial\Omega'} v\right)$$

and hence for all $x \in \bar\Omega'$ there holds

$$u(x) \le \max\left(0, \max_{\partial\Omega'}(u+\epsilon z)\right) \le \max\left(0, \max_{\partial\Omega}(u+\epsilon z) + \epsilon\max_{\bar\Omega} z\right).$$

This holds for all $\epsilon > 0$ and hence the result follows. □

As in the parabolic case, a uniqueness result follows.

2.2.7 Corollary. *Under the assumptions of Theorem 2.2.6 let $u, w \in C^2(\Omega)\cap C^0(\bar\Omega)$ satisfy*

$$Lu = Lw \text{ in } \Omega$$
$$u = w \text{ on } \partial\Omega.$$

Then $u = w$.

Strong maximum principles

In many situations the weak maximum principle is enough to deduce first a priori estimates. However, in some situations it is useful to know that the maximum can not be attained in the interior, unless the function is constant. This is the statement of the *strong maximum principle*. The crucial lemma in the parabolic case is the *propagation of positivity*, which is of independent interest, since it nicely demonstrates the diffusive effect of the heat equation, namely that heat tends to "spread out". We follow [11], with few adjustments which are due to the fact that we have not proven the weak maximum principle for general domains.

2.2.8 Lemma (Propagation of positivity)**.** *Let $n \in \mathbb{N}$, $\alpha, r, t_0 > 0$, $x_0 \in \mathbb{R}^n$ and*

$$Q = (t_0, t_0 + \alpha r^2) \times B_r(x_0) \subset \mathbb{R}^{n+1}.$$

Let $P = a^{ij}\partial^2_{ij} + b^i\partial_i + d - \partial_t$ be a linear uniformly parabolic operator in a neighborhood of Q with

$$\forall (t,x) \in \bar Q\ \forall (\xi_i) \in \mathbb{R}^n\colon \lambda|\xi|^2 \le a^{ij}(t,x)\xi_i\xi_j \le \Lambda|\xi|^2$$

for positive constants $\lambda < \Lambda$ and also suppose in $\bar Q$ that

$$|b| + |d| \le \Lambda, \quad d \le 0.$$

Let $u \in C^{1;2}(\bar Q)$ satisfy

$$u \ge 0, \quad Pu \le 0$$

and

$$\exists h > 0\ \exists 0 < \epsilon < \frac{1}{2}\ \forall x \in B_{\epsilon r}(x_0)\colon u(t_0, x) \ge h.$$

Then there exists a positive constant $\kappa = \kappa(\alpha, \lambda, \Lambda, r)$, such that

$$\forall x \in B_{\frac{r}{2}}(x_0)\colon u(t_0 + \alpha r^2, x) \ge \frac{\epsilon^\kappa}{2}h.$$

Proof. We may assume $t_0 = 0$ and $x_0 = 0$. The idea is to construct a *barrier* for u in some domain $\tilde{Q}$, which we will call χ. It must be constructed to satisfy $P\chi > 0$ in $\tilde{Q}$ and $\chi \leq u$ on a suitable piece of the boundary $\partial\tilde{Q}$.[5] Then we will conclude $\chi \leq u$[6]. If χ is good enough, we can conclude the desired estimate for u at time αr^2.

Since we have information about u on the piece $\{0\} \times B_{\epsilon r}(0)$ and want to obtain information on a bigger ball at later time, the most natural domain $\tilde{Q}$ is a trapezoid,

$$\tilde{Q} = \{(t,x) \in \mathbb{R}^{n+1} \colon t \in (0, \alpha r^2), |x|^2 < \varphi(t)\} \subset Q,$$

where

$$\varphi(t) = \frac{1-\epsilon^2}{\alpha} t + \epsilon^2 r^2.$$

At positive times, the function u is only known to be non-negative, hence we have no choice for the barrier, but to let it be zero on the boundary pieces $\{t\} \times \partial B_{\varphi^{\frac{1}{2}}(t)}(0)$. However at the time αr^2 we want the barrier to be positive in $\{\alpha r^2\} \times B_{\frac{r}{2}}(0)$ and the simplest function to satisfy this would be a quadratic one,

$$\psi(t,x) = (\varphi(t) - |x|^2).$$

This could be a first guess. Let us see what it gives. The first thing we have to calculate is $P\psi$. We have

$$\psi_{,i}(t,x) = -2x_i, \quad \psi_{,ij}(t,x) = -2\delta_{ij}, \quad \dot{\psi} = \frac{1-\epsilon^2}{\alpha}.$$

Hence

$$P\psi(t,x) = -2a^{ij}(t,x)\delta_{ij} - 2b^i(t,x)x_i + d\psi(t,x) - \frac{1-\epsilon^2}{\alpha}.$$

This does not seem promising yet, since the term $-2a^{ij}\delta_{ij}$ does not have a good sign. The function ψ is *too concave*. To make it more convex, we could square it. We get

$$\begin{aligned} P(\psi^2) &= 2\psi P\psi + 2a^{ij}\psi_{,i}\psi_{,j} - d\psi^2 \\ &= 2\psi P\psi + 8a^{ij}x_i x_j - d\psi^2 \\ &\geq 8\lambda|x|^2 - 4\psi \operatorname{tr}(a) - 4\psi|b||x| - 2|d|\psi^2 - 2\psi\frac{1-\epsilon^2}{\alpha} \\ &\geq 8\lambda\varphi - c\psi, \end{aligned}$$

where $c = c(\alpha, \lambda, \Lambda, r)$.

This looks better, but we still have to absorb the term involving ψ. As often in the theory of the parabolic equations, one can exploit the t-direction[7]. In

[5] We have not defined the parabolic boundary for general domains in $\mathbb{R}^{n+1}$, so we must work around this.

[6] By a simple argument. The weak maximum principle is not needed here and we have not proven it for general domains $\tilde{Q}$.

[7] as we have already seen in the proof of the weak maximum principle.

order to produce a good term coming from the t-direction, we multiply ψ^2 by a (possibly heavily) decreasing function in t. It must only contain values, which κ is allowed to depend on, hence φ seems to be a good candidate. Hence for $q > 0$ we put

$$z = \varphi^{-q}\psi^2.$$

Then

$$\begin{aligned}
Pz &= a^{ij}z_{,ij} + b^i z_{,i} + dz - \dot{z} \\
&= \varphi^{-q}P(\psi^2) + q\varphi^{-q-1}\frac{1-\epsilon^2}{\alpha}\psi^2 \\
&\geq \varphi^{1-q}\left(8\lambda - c\frac{\psi}{\varphi} + q\frac{1-\epsilon^2}{\alpha}\frac{\psi^2}{\varphi^2}\right) \\
&\geq \varphi^{1-q}\left(8\lambda - c^2\frac{\delta}{2} + \left(q\frac{1-\epsilon^2}{\alpha} - \frac{1}{2\delta}\right)\frac{\psi^2}{\varphi^2}\right) \\
&> 0
\end{aligned}$$

for small δ and large q. Now we have to take care about the boundary values. In order to adjust z to be less than u on the bottom of the cylinder, we have to multiply it by some factor. Put

$$\chi = h(\epsilon r)^{2q-4}z.$$

Let us compare χ with u. By construction there holds

$$\chi(t,x) = 0 \leq u \quad \forall x \in \partial B_{\varphi^{\frac{1}{2}}(t)}(0).$$

Furthermore, for all $|x| \leq \epsilon r$,

$$\chi(0,x) = h(\epsilon r)^{2q-4}(\epsilon r)^{-2q}(\epsilon^2 r^2 - |x|^2)^2 \leq h \leq u(0,x).$$

At all other points in the closure of $\tilde{Q}$ it is not possible for $\chi - u$ to obtain a positive maximum, since at such points we would have $P(\chi - u) \leq 0$, in contradiction to $P\chi > 0$ and $Pu \leq 0$. We conclude

$$\chi \leq u$$

throughout $\tilde{Q}$, which implies at $t = \alpha r^2$ and for all $|x| < r/2$:

$$u(\alpha r^2, x) \geq h(\epsilon r)^{2q-4}r^{-2q}(r^2 - |x|^2)^2 \geq \frac{9}{16}h\epsilon^\kappa.$$

□

We deduce the strong maximum principle.

2.2.9 Theorem (Parabolic strong maximum principle). *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ a domain, $T > 0$ and $Q = (0,T) \times \Omega$. Let*

$$Pu = a^{ij}u_{,ij} + b^i u_{,i} + du - \dot{u}$$

be locally[8] *uniformly parabolic with locally bounded coefficients and $d \leq 0$. Suppose $u \in C^{1;2}(Q)$ satisfies $Pu \geq 0$ and*

$$\exists (t_0, x_0) \in Q\colon \sup_Q u = u(t_0, x_0) \geq 0,$$

[8] On each compact set the coefficients have this property.

then

$$u_{|(0,t_0]\times\Omega} \equiv \text{const}\,.$$

Proof. It suffices to prove the constancy on $(0,t_0)\times\Omega$. Let

$$M = u(t_0,x_0)$$

and suppose there exists $(t,x)\in(0,t_0)\times\Omega$ with

$$u(t,x) < M.$$

Let $\gamma\colon [0,1]\to\Omega$ be a curve from x to x_0 and

$$S = \{\sigma\in[0,1]\colon u(st_0+(1-s)t,\gamma(s)) < M \quad \forall 0\le s\le\sigma\}.$$

Then $S\subset[0,1]$ is an interval with $0\in S$. Furthermore, whenever $1>s_0\in S$, due to continuity of u we find $\epsilon>0$, such that $s_0+\epsilon\in S$.[9] We will now show that S is also closed, then we conclude that $S=[0,1]$ and we have a contradiction. Therefore let s_k be a sequence in S which converges to $s_\infty\in[0,1]$ from below and set

$$t_k = s_k t_0 + (1-s_k)t.$$

Choose a ball

$$B_r(\gamma(s_\infty))\subset\Omega$$

and k so large that

$$\gamma(s_\infty)\in B_{\frac{r}{2}}(\gamma(s_k)).$$

Due to continuity there exists $0<\epsilon<1/2$ and $h>0$ such that

$$(M-u)(t_k,y)\ge h>0 \quad \forall y\in B_{\epsilon r}(\gamma(s_k)).$$

Pick

$$\alpha = \frac{t_\infty - t_k}{r^2}$$

and deduce from Lemma 2.2.8 that

$$(M-u)>0$$

on $\{t_\infty\}\times B_{\frac{r}{2}}(\gamma(s_k))$. □

The elliptic strong maximum principle is a consequence of the parabolic one.

2.2.10 Theorem (Elliptic strong maximum principle)**.** *Let $n\in\mathbb{N}$, $\Omega\subset\mathbb{R}^n$ be a domain and*

$$Lu = a^{ij}u_{,ij} + b^i u_{,i} + du$$

be locally uniformly elliptic with locally bounded coefficients b^i, d and $d\le 0$. Let $u\in C^2(\Omega)$ and $Lu\ge 0$, then u does not attain a non-negative maximum in Ω, unless u is constant.

[9] S is open in $[0,1]$.

Proof. Suppose the set

$$A = \{x \in \Omega \colon u(x) = \sup_{\Omega} u\}$$

is not empty. Due to continuity A is certainly closed. We prove that A is open. Let $x_0 \in A$ and $B_r(x_0) \subset \Omega$. Let $Q = (0,1) \times B_r(x_0)$ and set for all $(t,x) \in [0,1] \times \Omega$

$$v(t,x) = u(x).$$

Then $v \in C^2(\bar{Q})$ and $P = L - \partial_t$ is a linear uniformly parabolic operator in $\bar{Q}$. We have

$$Pv = Lv - \dot{v} = Lu \geq 0.$$

Furthermore for all t,

$$v(t,x_0) = \max_{\bar{Q}} v$$

and hence v is constant on $\{t\} \times B_r(x_0)$, which means that u is constant on $B_r(x_0)$. This proves that A is open and hence $A = \Omega$. □

The Hopf lemma

We prove the Hopf boundary point lemma for parabolic equations. Similar versions can be found in [11, Lemma II.2.8] and [4, Lemma 2.7.4].

2.2.11 Lemma. *Let $n \in \mathbb{N}$, $T > 0$, $z \in \mathbb{R}^n$, $x_0 \in \partial B_r(z)$, $0 < t_0 < T$ and*

$$Q = (0,T) \times B_r(z).$$

Let

$$P = a^{ij}\partial^2_{ij} + b^i\partial_i + d - \partial_t$$

be a linear uniformly parabolic operator in Q with bounded coefficients b^i, d and $d \leq 0$. Let $u \in C^{1;2}(Q) \cap C^1(\bar{Q})$ satisfy

$$Pu \geq 0, \quad u(t_0,x_0) \geq 0$$

and

$$\forall (t,y) \in (0,T) \times \bar{B}_r(z) \backslash \{(t_0,x_0)\} \colon u(t_0,x_0) > u(t,y).$$

Then there holds

$$\frac{\partial u}{\partial \nu}(t_0,x_0) > 0,$$

where ν denotes the exterior normal to $\partial \bar{B}_r(z)$ at x_0.

Proof. Let $0 < \rho < r$. In $A \equiv \bar{B}_r(z) \backslash B_\rho(z)$ define

$$h(x) = e^{-\alpha|x-z|^2} - e^{-\alpha r^2}, \quad \alpha > 0.$$

Then in A we have for suitable $\lambda > 0$,

$$\begin{aligned} Ph(x) &= e^{-\alpha|x-z|^2}(4\alpha^2 a^{ij}(x_i - z_i)(x_j - z_j) - 2\alpha a^{ij}\delta_{ij} - 2\alpha b^i(x_i - z_i)) + dh \\ &\geq e^{-\alpha|x-z|^2}(4\alpha^2\lambda|x-z|^2 - 2\alpha(a^i_i + |b||x-z|) - |d|) \\ &> 0, \end{aligned}$$

if α is large enough. Since for all $0 < \delta < t_0$ there holds

$$u_{|[\delta,t_0]\times\partial B_\rho(z)\cup\{\delta\}\times A} < u(t_0, x_0),$$

we find $\epsilon > 0$, such that in $[\delta, t_0] \times \partial B_\rho(z) \cup \{\delta\} \times A$ there holds

$$w \equiv u - u(x_0, t_0) + \epsilon h \leq 0.$$

Furthermore in $[\delta, t_0] \times \operatorname{int}(A)$ we have

$$Pw = Pu - u(t_0, x_0)d + \epsilon Ph > 0.$$

Thus we conclude $w \leq 0$ from the weak maximum principle, Theorem 2.2.2. Since $w(t_0, x_0) = 0$, we obtain

$$0 \leq \frac{\partial w}{\partial \nu}(t_0, x_0) = \frac{\partial u}{\partial \nu}(t_0, x_0) + \epsilon \frac{\partial h}{\partial \nu}(t_0, x_0),$$

from which the claim follows. □

The elliptic version, originally due to Eberhard Hopf [7], is suggested as an exercise.

2.2.12 Lemma (Eberhard Hopf). *Let $n \in \mathbb{N}$, $B \subset \mathbb{R}^n$ be a ball and $x_0 \in \partial B$. Let*

$$L = a^{ij}\partial^2_{ij} + b^i\partial_i + d$$

be a linear uniformly elliptic operator in B with bounded coefficients b^i, d and $d \leq 0$. Let $u \in C^2(B) \cap C^1(\bar{B})$ satisfy

$$Lu \geq 0, \quad u(x_0) \geq 0$$

and

$$\forall x \in B \colon u(x) < u(x_0).$$

Then there holds

$$\frac{\partial u}{\partial \nu}(x_0) > 0,$$

where ν is the outer normal to B in x_0.

A corollary of the Hopf lemma is a uniqueness result for the Neumann problem, the proof of which is an exercise. First we need another definition.

2.2.13 Definition (Interior ball condition). Let $\Omega \subset \mathbb{R}^n$ be open. For every boundary point $x_0 \in \partial\Omega$ we say that Ω satisfies an *interior ball condition at x_0*, if there exists a positive number r and a ball $B_r(x)$ such that

$$B_r(x) \subset \Omega, \quad \bar{B}_r(x) \cap \partial\Omega = \{x_0\}.$$

We say Ω satisfies an *interior ball condition*, if Ω satisfies an interior ball condition at every $x_0 \in \partial\Omega$.

2.2.14 Exercise. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain that satisfies and interior ball condition. Let

$$L = a^{ij}\partial^2_{ij} + b^i\partial_i + d$$

be a linear uniformly elliptic operator with bounded coefficients b^i, d and $d \leq 0$. Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfy the Neumann problem

$$Lu = 0 \text{ in } \Omega$$
$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega,$$

then u is constant in Ω.

2.3 Comparison principles for fully nonlinear operators

The maximum principle is not restricted to linear equations. In this section we will employ the linear case to prove comparison principles for fully nonlinear equations. First of all we have to say, when a fully nonlinear operator is elliptic or parabolic. A good orientation for this section is [5].

Fully nonlinear elliptic and parabolic operators

2.3.1 Definition (Elliptic operators)**.** Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ open.

(i) Let
$$\Gamma \subset \mathbb{R}^{n^2} \times \mathbb{R}^n \times \mathbb{R} \times \Omega.$$
A *partial differential operator of second order in* Ω is a map
$$\mathcal{L}_F \colon \mathcal{A} \subset C^2(\Omega) \to \mathbb{R}^\Omega$$
$$u \mapsto F(D^2u, Du, u, \cdot),$$
where $F = F(r, p, z, x)$ is a map $F \colon \Gamma \to \mathbb{R}$ and $\mathcal{A}$ is the set of F-*admissable* functions, i.e.
$$\forall u \in \mathcal{A}\ \forall x \in \Omega \colon (D^2u(x), Du(x), u(x), x) \in \Gamma.$$
$\mathcal{L}_F$ is called *elliptic in* $u \in \mathcal{A}$, if
$$\left(F^{ij}(D^2u(x), Du(x), u(x), x)\right) := \left(\frac{\partial F}{\partial r_{ij}}(D^2u(x), Du(x), u(x), x)\right)_{\text{sym}}$$
exists and is positive definite for all $x \in \Omega$. For a set $\mathcal{S} \subset \mathcal{A}$, $\mathcal{L}_F$ is called *elliptic operator in* $\mathcal{S}$, if $\mathcal{L}_F$ is elliptic in all $u \in \mathcal{S}$.[10]

(ii) Let $\mathcal{S} \subset \mathcal{A}$. $\mathcal{L}_F$ is called *strictly elliptic in* $\mathcal{S}$, if
$$\exists \lambda > 0\ \forall u \in \mathcal{S}\ \forall (\xi_i) \in \mathbb{R}^n \colon F^{ij}(D^2u, Du, u, \cdot)\xi_i\xi_j \geq \lambda|\xi|^2$$
and *uniformly elliptic in* S, if
$$\exists 0 < \lambda < \Lambda\ \forall u \in \mathcal{S}\ \forall (\xi_i) \in \mathbb{R}^n \colon \lambda|\xi|^2 \leq F^{ij}(D^2u, Du, u, \cdot)\xi_i\xi_j \leq \Lambda|\xi|^2.$$

[10] For a matrix A, A_{sym} denotes its symmetrisation $\frac{1}{2}(A + A^t)$.

2.3.2 Example. (i) Every linear elliptic operator in an open set Ω is an elliptic operator in $C^2(\Omega)$, since

$$Lu = a^{ij}u_{,ij} + b^i u_{,i} + du = F(D^2u, Du, u, \cdot)$$

with

$$F(r,p,z,x) = a^{ij}(x)r_{ij} + b^i(x)p_i + d(x)z.$$

(ii) The *Monge-Ampère-equation* is

$$\det(D^2u) = f$$

with a function $f \in C^0(\Omega)$. The corresponding differential operator is then

$$\mathcal{L}_F(u) = F(D^2u) = \det(D^2u).$$

If r is invertible, from (1.6) we obtain

$$\frac{\partial F}{\partial r_{ij}}(r) = \left((\det r)r^{ij}\right)_{\mathrm{sym}},$$

where $r^{-1} = (r^{ij})$. Hence $\mathcal{L}_F$ is elliptic all all strictly convex u.

2.3.3 Exercise. The *equation of prescribed mean curvature* is

$$H(D^2u, Du, u, \cdot) := \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f.$$ [11]

Prove that on each set

$$\mathcal{A}_c = \{u \in C^2(\Omega) \colon |\nabla u|^2 \le c\}$$

$\mathcal{L}_H$ is uniformly elliptic.

We have a similar definition for the parabolic case.

2.3.4 Definition (Parabolic operators)**.** Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open, $0 < T \le \infty$ and $Q = (0,T) \times \Omega$

(i) Let

$$\Gamma \subset \mathbb{R}^{n^2} \times \mathbb{R}^n \times \mathbb{R} \times Q.$$

A partial differential operator of second order in Q of the special form

$$\mathcal{P}_F \colon \mathcal{A} \subset C^{1;2}(Q) \to \mathbb{R}^Q$$
$$u \mapsto F(D_x^2u, D_xu, u, \cdot) - \dot{u},$$

where $F = F(r,p,z,t,x)$ is a map $F \colon \Gamma \to \mathbb{R}$ and $\mathcal{A}$ is the set of F-*admissable* functions, i.e.

$$\forall u \in \mathcal{A}\ \forall (t,x) \in Q \colon (D_x^2u(t,x), D_xu(t,x), u(t,x), t, x) \in \Gamma,$$

[11] Recall the relation between Du and ∇u, cf. Definition 1.4.5.

is called *parabolic in* $u \in \mathcal{A}$, if

$$F^{ij} := \left(\frac{\partial F}{\partial r_{ij}} (D_x^2 u(t,x), D_x u(t,x), u(t,x), t, x) \right)_{\text{sym}}$$

exists and is positive definite for all $(t,x) \in Q$. For a set $\mathcal{S} \subset \mathcal{A}$, $\mathcal{P}_F$ is called *parabolic operator in* $\mathcal{S}$, if $\mathcal{P}_F$ is parabolic in all $u \in \mathcal{S}$.

(ii) Let $\mathcal{S} \subset \mathcal{A}$. $\mathcal{P}_F$ is called *strictly parabolic in* $\mathcal{S}$, if

$$\exists \lambda > 0 \ \forall u \in \mathcal{S} \ \forall (\xi_i) \in \mathbb{R}^n \colon F^{ij}(D_x^2 u, D_x u, u, \cdot)\xi_i \xi_j \geq \lambda |\xi|^2$$

and *uniformly parabolic in* S, if

$$\exists 0 < \lambda < \Lambda \ \forall u \in \mathcal{S} \ \forall (\xi_i) \in \mathbb{R}^n \colon \lambda |\xi|^2 \leq F^{ij}(D_x^2 u, D_x u, u, \cdot)\xi_i \xi_j \leq \Lambda |\xi|^2.$$

2.3.5 Example (Mean curvature flow). The differential equation

$$\partial_t u = \sqrt{1 + |\nabla u|^2} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right)$$

is called the (graphical) *mean curvature flow.* As in Exercise 2.3.3 we can check that the mean curvature flow is uniformly parabolic on each set of functions with bounded spatial gradient.

Comparison principles

Now we prove versions of the maximum principle for general (fully nonlinear) elliptic and parabolic operators. In this context they are called *comparison principles.* We start with the elliptic case.

2.3.6 Theorem (Elliptic comparison principle)**.** *Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ open and bounded,*

$$\Gamma \subset \mathbb{R}^{n^2} \times \mathbb{R}^n \times \mathbb{R} \times \Omega$$

and $F \in \mathbb{R}^\Gamma$ continuously differentiable in its first three variables. Let $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and let $\mathcal{L}_F$ be a strictly elliptic operator on

$$\mathcal{S} = \{\tau u + (1-\tau)v \colon \tau \in [0,1]\}$$

with

$$\exists c > 0 \ \forall w \in \mathcal{S} \ \forall x \in \Omega \colon \left| \frac{\partial F}{\partial p}(D^2 w(x), Dw(x), w(x), x) \right| \leq c$$

and

$$\forall w \in \mathcal{S} \colon \frac{\partial F}{\partial z}(D^2 w, Dw, w, \cdot) \leq 0.$$

Suppose u, v satisfy

$$\begin{aligned} F(D^2 u, Du, u, \cdot) &\geq F(D^2 v, Dv, v, \cdot) \quad && \text{in } \Omega, \\ u &\leq v && \text{on } \partial\Omega \end{aligned}$$

then there holds

$$u \leq v \quad \text{in } \Omega.$$

Proof. Define

$$\chi = u - v.$$

There holds

$$\begin{aligned}
0 &\leq F(D^2u, Du, u, \cdot) - F(D^2v, Dv, v, \cdot) \\
&= \int_0^1 \frac{d}{d\tau} F(\tau D^2u + (1-\tau)D^2v, \tau Du + (1-\tau)Dv, \tau u + (1-\tau)v, \cdot)\ d\tau \\
&= \int_0^1 F^{ij}\chi_{ij} + \int_0^1 \frac{\partial F}{\partial p_i}\chi_i + \int_0^1 \frac{\partial F}{\partial z}\chi \\
&= a^{ij}\chi_{,ij} + b^{,i}\chi_i + d\chi
\end{aligned}$$

with

$$a^{ij} = \int_0^1 F^{ij}(\tau D^2u + (1-\tau)D^2v, \tau Du + (1-\tau)Dv, \tau u + (1-\tau)v, \cdot)\ d\tau$$

and similarly for b^i and d. Thus χ satisfies the linear problem

$$\begin{aligned}
L\chi \equiv a^{ij}\chi_{,ij} + b^i\chi_{,i} + d\chi &\geq 0 \quad \text{in } \Omega, \\
\chi &\leq 0 \quad \text{on } \partial\Omega
\end{aligned}$$

with positive definite (a^{ij}) and

$$\frac{|b|}{\lambda} \leq c, \quad d \leq 0.$$

The weak maximum principle, Theorem 2.2.6, gives

$$\chi \leq 0$$

in all of Ω. □

The parabolic case is similar.

2.3.7 Theorem (Parabolic comparison principle)**.** *Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ open and bounded, $0 < T \leq \infty$, $Q = (0,T) \times \Omega$,*

$$\Gamma \subset \mathbb{R}^{n^2} \times \mathbb{R}^n \times \mathbb{R} \times Q$$

and $F \in \mathbb{R}^\Gamma$ continuously differentiable in its first three variables. Let $u, v \in C^{1;2}(Q) \cap C^0(\bar{Q})$ and let $\mathcal{P}_F$ be a strictly parabolic operator on

$$\mathcal{S} = \{\tau u + (1-\tau)v \colon \tau \in [0,1]\}$$

with

$$\forall w \in \mathcal{S} \colon \frac{\partial F}{\partial z}(D_x^2 w, D_x w, w, \cdot) \leq 0.$$

Suppose u, v satisfy

$$\begin{aligned}
F(D_x^2u, D_xu, u, \cdot) - \dot{u} &\geq F(D_x^2v, D_xv, v, \cdot) - \dot{v} \quad \text{in } Q, \\
u &\leq v \quad \text{on } \partial_p Q
\end{aligned}$$

then there holds

$$u \leq v \quad \text{in } Q.$$

Proof.

$$\chi = u - v.$$

There holds

$$\begin{aligned} 0 &\le F(D_x^2 u, D_x u, u, \cdot) - \dot u - F(D_x^2 v, D_x v, v, \cdot) + \dot v \\ &= \int_0^1 \frac{d}{d\tau}(F(\tau D_x^2 u + (1-\tau) D_x^2 v, \tau D_x u + (1-\tau) D_x v, \tau u + (1-\tau) v, \cdot)) \\ &\quad - (\tau \dot u + (1-\tau)\dot v)\ d\tau \\ &= \int_0^1 F^{ij}\chi_{ij} + \int_0^1 \frac{\partial F}{\partial p_i}\chi_i + \int_0^1 \frac{\partial F}{\partial z}\chi - \dot\chi \\ &= a^{ij}\chi_{,ij} + b^i\chi_{,i} + d\chi - \dot\chi \end{aligned}$$

with

$$a^{ij} = \int_0^1 F^{ij}(tD_x^2 u + (1-t)D_x^2 v, tD_x u + (1-t)D_x v, tu + (1-t)v, \cdot)\ dt$$

and similarly for b^i and d. Thus χ satisfies the linear problem

$$\begin{aligned} P\chi \equiv a^{ij}\chi_{,ij} + b^i\chi_{,i} + d\chi - \dot\chi \ge 0 \quad &\text{in } Q, \\ \chi \le 0 \quad &\text{on } \partial Q \end{aligned}$$

with positive definite (a^{ij}) and $d \le 0$. The weak maximum principle, Theorem 2.2.2, gives

$$\chi \le 0$$

in all of Q. □

Chapter 3

Sobolev-Spaces

Until now we have obtained some uniqueness results for various kinds of PDE, e.g.

$$\begin{aligned} \Delta u &= f \text{ in } \Omega \\ u &= \varphi \text{ on } \partial\Omega, \end{aligned}$$

but we have not said anything about actual existence of a solution. Even if f is smooth, it can be difficult to prove existence of a smooth solution directly, since the spaces $C^k(\bar{\Omega})$ are relatively small for this purpose. So the strategy is to widen the space, in which we look for solutions. Hence we leave the class of differentiable functions and look instead at the space of *weakly differentiable functions*, the so-called *Sobolev-spaces* $W^{k,p}(\Omega)$. Here one can use Banach- or Hilbertspace methods to get existence of a weak solution relatively easy. Afterwards we will show how smooth this weak solution actually is, dependening on the right hand side f.

The present chapter is devoted to provide the necessary theory of the Sobolev spaces. Good sources for this chapter are [5, Ch. 7] and [17].

3.1 Elements of functional analysis

The theory of weak solutions to partial differential equations requires some basic knowledge in functional analysis, which was promised not to be required to follow this course. Hence this section is devoted to provide the results we need. During the following weeks, this is a dynamic section, which means that it grows while we are already talking about Sobolev spaces. This strategy has two advantages: Firstly you will see the theory of functional analysis "in action" right away and secondly the various results will not be scattered around within the rest of this chapter, but will be thoroughly collected in this one section.

Mollifiers and smooth approximation

The following construction is an extremely useful tool to carry over properties of smooth functions to less smooth functions.

3.1.1 Definition (Mollifier)**.** Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ open.

(i) A *mollifier* is a non-negative function $\eta \in C_c^\infty(\mathbb{R}^n)$ with

$$\operatorname{supp}\eta \Subset \bar{B}_1(0), \qquad \int_{\mathbb{R}^n} \eta = 1.$$

(ii) For a mollifier η, a function $u \in L^1_{\text{loc}}(\Omega)$, $\Omega' \Subset \Omega$ and $0 < \epsilon < \operatorname{dist}(\Omega', \partial\Omega)$ we define their *ϵ-convolution* by

$$u_\epsilon(x) = \int_{\mathbb{R}^n} \eta_\epsilon(x-y)u(y)\ dy \quad \forall x \in \Omega',$$

where

$$\eta_\epsilon(x) = \epsilon^{-n}\eta\left(\frac{x}{\epsilon}\right).$$

A special feature of the convolution is, as a rule of thumb, that it approximates a function locally as strongly as the function actually is. The proof of the following proposition is an exercise.

3.1.2 Proposition. *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be open, $\epsilon > 0$, η a mollifier and $u \in L^1_{\text{loc}}(\Omega)$. Then there hold:*

(i)

$$\forall \Omega' \Subset \Omega\ \forall \epsilon < \operatorname{dist}(\Omega', \partial\Omega)\colon u_\epsilon \in C^\infty(\Omega').$$

(ii) If $u \in C^k(\Omega)$ for $0 \le k < \infty$, then

$$\forall \Omega' \Subset \Omega\colon \lim_{\epsilon \to 0} |u_\epsilon - u|_{k,\Omega'} = 0.$$

(iii) If $u \in L^p_{\text{loc}}(\Omega)$, $p < \infty$, then

$$\forall \Omega' \Subset \Omega\colon \lim_{\epsilon \to 0} \|u_\epsilon - u\|_{p,\Omega'} = 0.$$

(iv) $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$, if $p < \infty$.

As an application we prove one of the most important tools in analysis, the *fundamental lemma of the calculus of variations.*

3.1.3 Lemma (Fundamental lemma of the calculus of variations)**.** *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open, $f \in L^1_{\text{loc}}(\Omega)$ and suppose*

$$\forall \varphi \in C_c^\infty(\Omega)\colon \int_\Omega f\varphi \ge 0.$$

Then there holds $f \ge 0$ almost everywhere in Ω.

Proof. We have to show that

$$\mathcal{L}^n(E) = \mathcal{L}^n(\{f < 0\}) = 0.$$

We may assume that $\bar{E} \subset \Omega$ is compact, otherwise consider a countable exhaustion of Ω by compactly contained sets. Since $\chi_E \in L^1(\Omega)$, there exist $g_n \in C_c^\infty(\Omega)$ with

$$g_n = \int_{\mathbb{R}^n} \eta_{\epsilon_n}(\cdot - y)\chi_E(y)\ dy \to \chi_E$$

in $L^1_{\text{loc}}(\Omega)$ and pointwise almost everywhere in E with a sequence $\epsilon_n \to 0$. Due to the Lebesgue convergence theorem we get

$$0 \leq \int_\Omega f g_n \to \int_\Omega f \chi_E.$$

Due to $f_{|E} < 0$ we obtain $\mathcal{L}^n(E) = 0$. □

Linear operators

We collect some basics about linear maps between normed spaces and follow [3, Sec. 2.7].

3.1.4 Proposition. *Let E and F be normed vector spaces over $\mathbb{K}$ and*

$$A \colon E \to F$$

be a linear map.[1] *Then A is continuous if and only if there exists a constant $c > 0$, such that*

$$\|Ax\| \leq c\|x\| \quad \forall x \in E.$$[2]

Proof. If such a constant exists, then

$$\|Ax - Ay\| \leq c\|x - y\|$$

and A is continuous. Now suppose A is continuous. If c does not exists, then there exists a sequence $x_n \in E$ such that

$$\|Ax_n\| > n\|x_n\|.$$

Thus

$$1 \leq \left\| A\left(\frac{x_n}{n\|x_n\|}\right)\right\| \to 0,$$

a contradiction. □

3.1.5 Definition. Let E and F be normed vector spaces over $\mathbb{K}$.

(i) Define $L(E,F)$ to be the $\mathbb{K}$-vector space of continuous linear maps from E to F. For $A \in L(E,F)$ define

$$\|A\|_{L(E,F)} = \inf\{c \geq 0 \colon \|Ax\| \leq c\|x\| \quad \forall x \in E\}.$$

(ii) We also write

$$E' = L(E, \mathbb{K}).$$

3.1.6 Exercise. Let E and F be normed vector spaces over $\mathbb{K}$.

[1] A linear map between normed spaces is often called linear *operator* and if $F = \mathbb{K}$ it is also called linear *functional*.

[2] If no ambiguities are possible, we do not distinguish the norms in E and F notationally.

(i) Prove that $(L(E,F), \|\cdot\|_{L(E,F)})$ is a normed vector space over $\mathbb{K}$, which is complete if F is complete.

(ii) There holds for all $A \in L(E,F)$:

$$\|A\| = \sup_{x\in E\setminus\{0\}} \frac{\|Ax\|}{\|x\|}.$$

(iii) Let (H,g) be an inner product space. Then there holds the *Cauchy-Schwarz inequality*

$$|g(x,y)| \leq \|x\|_g \|y\|_g.$$

The Riesz representation theorem

In Hilbert spaces H every continuous linear functional is given as a scalar product with a fixed $x \in H$. To prove this, we need a lemma.

3.1.7 Lemma (Projection onto closed subspaces)**.** *Let (H,g) be a Hilbert space over $\mathbb{R}$[3] and M a closed subspace. Then for all $x \in H$ there exist $y \in M$ and*

$$z \in M^{\perp} := \{z \in H : g(x,z) = 0 \ \forall x \in M\},$$

such that

$$x = y + z.$$

Proof. We may suppose $x \notin M$. Define

$$d = \operatorname{dist}(x, M)$$

and let $(y_n)_{n\in\mathbb{N}}$ be a minimizing sequence, i.e.

$$d(x, y_n) \to d.$$

We have

$$\begin{aligned}\|y_n - y_m\|_g^2 &= 2\|x - y_n\|_g^2 + 2\|x - y_m\|_g^2 - 4\left\|x - \frac{1}{2}(y_n + y_m)\right\|_g^2 \\ &\leq 2\|x - y_n\|_g^2 + 2\|x - y_m\|_g^2 - 4d^2 \\ &\to 0,\end{aligned}$$

as $n, m \to \infty$. Hence (y_n) is a Cauchy sequence which has a limit $y \in M$. Put

$$z = x - y,$$

then for all $y' \in M$ there holds

$$0 = \frac{d}{dt}\|x - (y + ty')\|_{g,|t=0}^2 = -2g(z, y')$$

and hence $z \in M^{\perp}$. ☐

[3] only for simplicity

3.1.8 Theorem (Riesz representation theorem). *Let (H, g) be a Hilbert space over $\mathbb{R}$. Then the map*

$$\begin{aligned} J \colon H &\to H' \\ y &\mapsto g(\cdot, y) \end{aligned}$$

is a norm preserving linear bijection.

Proof. J maps to H', since $J(y)$ is linear and for all $x \in H$

$$|J(y)x| \leq |g(x, y)| \leq \|y\|_g \|x\|_g.$$

J is obviously linear and we have

$$\|J(y)\|_{H'} \leq \|y\|.$$

From $J(y)y = \|y\|_g^2$ we also obtain

$$\|J(y)\|_{H'} \geq \|y\|_g.$$

Hence J is norm preserving and thus injective. It remains to prove the surjectivity. Hence let $\psi \in H'$. If $\psi = 0$ we take $0 \in H$. Otherwise pick $z \in H$ with the properties

$$\|z\| = 1, \quad g(z, y) = 0 \quad \forall y \in \ker(\psi).$$

For all $x \in H$ we have

$$x - \frac{\psi(x)}{\psi(z)} z \in \ker(\psi)$$

and thus

$$g\left(x, \psi(z) z\right) = \psi(x) \quad \forall x \in H.$$

□

Weak compactness in Hilbert spaces

The closed unit ball in $\mathbb{R}^n$ is compact. But infinite dimensional Banach spaces this is not true anymore. However, in this situation it is *weakly compact*, as we will prove in this subsection.

3.1.9 Definition (Weak convergence). Let $(E, \|\cdot\|)$ be a normed vector space over $\mathbb{K}$. A sequence $(x_n)_{n \in \mathbb{N}}$ *converges weakly* to $x \in E$, if

$$\forall \phi \in E' \colon \phi(x_n) \to \phi(x).$$

In this case we write

$$x_n \rightharpoonup x.$$

Before we can prove the weak compactness of bounded sets in Hilbert spaces, we recall the following very useful construction.

3.1.10 Lemma (Cantor's diagonal sequence). *Let A, B be sets, M a metric space and*

$$g \colon A \times B \to M$$

a map. Let $(x_n)_{n \in \mathbb{N}}$ and $(y_k)_{k \in \mathbb{N}}$ be sequences in A resp. B, such that for all $k \in \mathbb{N}$ the sequence $(g(x_n, y_k))_{n \in \mathbb{N}}$ has a convergent subsequence. Then there exists a subsequence $(x_i)_{i \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$, such that

$$\forall k \in \mathbb{N}\ \exists \alpha_k \in M \colon \lim_{i \to \infty} g(x_i, y_k) = \alpha_k.$$

Proof. We construct a sequence of subsequences of $(x_n)_{n\in\mathbb{N}}$ inductively. Since $(g(x_n, y_1))_{n\in\mathbb{N}}$ has a convergent subsequence, there exists a first subsequence $(x_{n_j^1})_{j\in\mathbb{N}}$ and $\alpha_1 \in M$, such that

$$\lim_{j\to\infty} g\left(x_{n_j^1}, y_1\right) \to \alpha_1.$$

Let m subsequences $(x_{n_j^1})_{j\in\mathbb{N}}, \ldots, (x_{n_j^m})_{j\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and $\alpha_1, \ldots, \alpha_m$ be constructed, such that for all all $1 \le l \le m-1$, $(n_j^{l+1})_{j\in\mathbb{N}}$ is a subsequence of $(n_j^l)_{j\in\mathbb{N}}$ and all $1 \le l \le m$ there holds

$$g(x_{n_j^l}, y_l) \to \alpha_l.$$

The sequence $(g(x_{n_j^m}, y_{m+1}))_{j\in\mathbb{N}}$ contains a convergent subsequence

$$\lim_{j\to\infty} g\left(x_{n_j^{m+1}}, y_{m+1}\right) = \alpha_{m+1}.$$

We have constructed a sequence of subsequences $((x_{n_j^m})_{j\in\mathbb{N}})_{m\in\mathbb{N}}$ with the properties that $(n_j^{m+1})_{j\in\mathbb{N}}$ is a subsequence of $(n_j^m)_{j\in\mathbb{N}}$ and

$$\forall m \in \mathbb{N}\colon \lim_{j\to\infty} g(x_{n_j^m}, y_m) = \alpha_m. \tag{3.1}$$

Then the *diagonal sequence*

$$(x_i)_{i\in\mathbb{N}} = \left(x_{n_i^i}\right)_{i\in\mathbb{N}}$$

has the property

$$\forall k \in \mathbb{N}\colon \lim_{i\to\infty} g(x_i, y_k) = \alpha_k,$$

since for every $k \in \mathbb{N}$, the sequence $(x_{n_i^i})_{i\ge k}$ is a subsequence of $(x_{n_i^k})_{i\in\mathbb{N}}$ and the latter satisfies (3.1). $\square$

3.1.11 Theorem. *Let (H, g) be a Hilbert space over $\mathbb{R}$, then every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in H has a weakly convergent subsequence.*

Proof. By the Riesz representation theorem it suffices to prove:

$$\exists x \in H\ \forall y \in H\colon g(x_n, y) \to g(x, y).$$

First suppose that there exists a countable dense set $\{y_k\}_{k\in\mathbb{N}}$ in H. By Cantor's diagonal method we obtain a subsequence $(x_i)_{i\in\mathbb{N}}$ of (x_n) with the property

$$\forall k \in \mathbb{N}\ \exists \alpha_k \in \mathbb{R}\colon \lim_{i\to\infty} g(x_i, y_k) = \alpha_k.$$

Define

$$\psi(y_k) = \lim_{i\to\infty} g(x_i, y_k) = \alpha_k,$$

then ψ is a continuous linear map on $\operatorname{span}(y_k)_{k\in\mathbb{N}}$ due to the boundedness of $(x_i)_{i\in\mathbb{N}}$. Hence it may be extended to a bounded linear functional $\psi \in H'$, for which we find $x \in H$ with

$$\forall y \in H\colon \psi(y) = g(y, x).$$

Hence for all $y \in H$

$$\begin{aligned}|g(x_i,y)-g(x,y)| &\le |g(x_i,y)-g(x_i,y_k)|+|g(x_i,y_k)-g(x,y_k)| \\ &\quad + |g(x,y_k)-g(x,y)| \\ &\le (c+\|x\|)\|y_k-y\|+|g(x_i,y_k)-g(x,y_k)|.\end{aligned}$$

Choosing $\|y-y_k\|$ small and then i large gives

$$\forall y \in H\colon g(x_i,y)\to g(x,y).$$

If there is no countable dense subset, first apply the previous result to

$$H_0=\overline{\operatorname{span}(x_n)_{n\in\mathbb{N}}}$$

and obtain a subsequence (x_i) and $x \in H_0$ such that

$$\forall y \in H_0\colon g(x_i,y)\to g(x,y).$$

For arbitrary $y \in H$ let, according to Lemma 3.1.7,

$$y=y_1+y_0,$$

where $y_0 \in H_0$ and $y_1 \in H_0^{\perp}$. Then

$$g(x_i,y)=g(x_i,y_0)\to g(x,y_0)=g(x,y).$$

□

We will also need the following fact.

3.1.12 Proposition (Weak lower semicontinuity). *Let (H,g) be a Hilbert space and suppose $x_n \rightharpoonup x$. Then*

$$\|x\|_g \le \liminf_{n\to\infty}\|x_n\|_g.$$

Proof.

$$\|x\|_g^2=g(x,x)=\liminf_{n\to\infty} g(x_n,x)\le \liminf_{n\to\infty}\|x_n\|_g\|x\|_g.$$

□

Fredholm alternative in Hilbert spaces

From elementary linear algebra we know that a linear map from $\mathbb{R}^n$ to itself is injective if and only if it is surjective. In infinite dimensional space this is not true in general, as can be seen from the shift operator on $l^\infty(\mathbb{R})$

$$(x_n)_{n\in\mathbb{N}}\mapsto(0,x_1,x_2,\dots).$$

However, for certain "small" perturbations of the identity this is still true and we will prove this now. For simplicity we restrict to Hilbert spaces again.

3.1.13 Definition. Let E and F be normed vector spaces over $\mathbb{K}$ and

$$A\colon E\to F$$

be a linear map. A is called *compact*, if for every bounded sequence $(x_n)_{n\in\mathbb{N}}$, a subsequence of $(Ax_n)_{n\in\mathbb{N}}$ converges in F.

3.1.14 Theorem (Fredholm alternative)**.** *Let (H, g) be a real Hilbert space and $T\colon H \to H$ compact. Then $I - T$ is injective if and only if $I - T$ is surjective. In this case $(I - T)^{-1}$ is continuous.*

Proof. Let $S := I - T$. We write $\|\cdot\| = \|\cdot\|_g$. The proof contains four steps.

(i)

$$\exists c > 0 \ \forall x \in H\colon \operatorname{dist}(x, \ker(S)) \leq c\|Sx\|, \tag{3.2}$$

since if (3.2) was wrong, then

$$\exists x_n \in H\colon d_n = \operatorname{dist}(x_n, \ker(S)) > n\|Sx_n\|$$

w.l.o.g. $\|Sx_n\| = 1$, such that $d_n > n$. Choose $y_n \in \ker(S)$ such that

$$d_n \leq \|x_n - y_n\| \leq 2d_n. \tag{3.3}$$

and define

$$z_n := \frac{x_n - y_n}{\|x_n - y_n\|}.$$

Then

$$\|z_n\| = 1, \quad \|Sz_n\| = \frac{\|Sx_n\|}{\|x_n - y_n\|} \leq \frac{1}{d_n} \to 0.$$

$Sz_n = z_n - Tz_n$ and T is compact, we get

$$z_n \to y_0$$

for a subsequence and hence

$$Sz_n \to Sy_0 = 0,$$

which implies $y_0 \in \ker(S)$, which is a contradiction, since

$$\begin{aligned}\operatorname{dist}(z_n, \ker(S)) &= \inf_{y \in \ker(S)} \left\| \frac{x_n - y_n}{\|x_n - y_n\|} - y \right\| \\ &= \inf_{y \in \ker(S)} \frac{1}{\|x_n - y_n\|}\|x_n - y\| = \frac{d_n}{\|x_n - y_n\|} \geq \frac{1}{2},\end{aligned}$$

by (3.3).

(ii) The image of S, $R = R(S)$, is closed: Suppose

$$Sx_n \to y \in H.$$

Step (i) implies that $d_n \leq c\|Sx_n\|$ and the latter is bounded. Choose $y_n \in \ker(S)$ as in (3.3). Then

$$w_n = x_n - y_n$$

is bounded and

$$Sw_n = Sx_n \to y.$$

Since T is compact, there holds $Tw_n \to w_0$ for a subsequence and hence

$$w_n \to y + w_0$$

and

$$S(y + w_0) = y.$$

(iii)

$$\ker(S) = \{0\} \quad \Rightarrow \quad R = R(S) = H.$$

Suppose the claim was wrong and define $R_j := S^j(H)$. Then $R_j \subset R_{j-1}$. Consider

$$S\colon R_j \to R_j.$$

Then by step (ii) R_{j+1} is closed. We claim

$$\exists k \in \mathbb{N}\ \forall j \geq k\colon R_j = R_k.$$

Otherwise choose orthogonal elements

$$x_n \in R_n\colon \|x_n\| = 1, \quad x_n \perp R_{n+1}.$$

Let $n > m$, then

$$Tx_m - Tx_n = x_m + (-x_n - Sx_m + Sx_n)$$

and hence

$$\|Tx_m - Tx_n\| \geq 1,$$

which is in contradiction to the compactness of T. So let $y \in H$, then $S^k y \in R_k = R_{k+1}$, then

$$0 = S^k y - S^{k+1}x = S^k(y - Sx),$$

$$y = Sx$$

and thus S is surjective.

(iv)

$$R = H \quad \Rightarrow \quad \ker(S) = \{0\}.$$

The sequence $N_j = \ker(S^j)$ consists of closed subspaces

$$N_j \subset N_{j+1}, \quad j \geq 1.$$

We claim that

$$\exists k \in \mathbb{N}\ \forall j \geq k\colon N_j = N_k.$$

If the claim was wrong, then

$$\exists x_m \in N_m\colon \|x_m\| = 1, \quad x_m \perp N_{m-1}.$$

Let $m > n$ then, analogously to step (iii), we obtain a contradiction due to

$$Tx_m - Tx_n = x_m + (-x_n - Sx_m + Sx_n),$$

since $S(N_i) \subset N_{i-1}$. So suppose $R = H$, then for all k there holds $R(S^k) = H$. Hence

$$\forall y \in N_k\ \exists x \in H\colon 0 = S^k y = S^{2k}x.$$

Then

$$x \in N_{2k} = N_k,$$

hence $y = 0$ and

$$\ker(S) = N_1 \subset N_k = \{0\}.$$

□

Theorem of Lax-Milgram

We need a refinement of the Riesz representation theorem. For this we need a lemma, the proof of which is an exercise.

3.1.15 Exercise. Let (H, g) be a real Hilbert space and $T \in L(H, H)$ satisfy

$$\exists c > 0 \ \forall x \in H \colon \|x\|_g \leq c\|Tx\|_g.$$

Prove that $T(H) \subset H$ is a closed subspace.

3.1.16 Theorem (Lax-Milgram)**.** *Let (H, g) be a real Hilbert space and $B \colon H \times H \to \mathbb{R}$ be a bilinear form, which is bounded, i.e.*

$$\exists c > 0 \ \forall x, y \in H \colon |B(x, y)| \leq c\|x\|_g\|y\|_g$$

and coercive, i.e.

$$\exists \lambda > 0 \ \forall x \in H \colon B(x, x) \geq \lambda\|x\|_g^2.$$

Then for every $\psi \in H'$ there exists a unique $v \in H$, such that

$$B(\cdot, v) = \psi.$$

Proof. Let $w \in H$. By the Riesz representation theorem there exists a unique $Tw \in H$, such that

$$B(\cdot, w) = g(\cdot, Tw).$$

This defines a linear map $T \colon H \to H$. There holds

$$\|Tw\|_g^2 = B(Tw, w) \leq c\|w\|_g\|Tw\|_g$$

and hence $T \in L(H, H)$. Furthermore

$$\lambda\|w\|_g^2 \leq B(w, w) = g(w, Tw) \leq \|w\|_g\|Tw\|_g$$

and hence

$$\forall w \in H \colon \|Tw\|_g \geq \lambda\|w\|_g.$$

Hence T is injective and has closed range. Suppose $T(H) \neq H$. Then there exists an orthonormal element $z \in H$, i.e.

$$\forall w \in H \colon g(z, Tw) = 0.$$

Putting $w = z$ we obtain $B(z, z) = 0$ and hence $z = 0$. Thus T is bijective with continuous inverse.

Now let $\psi \in H'$ be given, and w be such that

$$g(\cdot, w) = \psi.$$

Set

$$v = T^{-1}w$$

and obtain

$$B(\cdot, v) = g(\cdot, w) = \psi.$$

□

Compactness in function spaces

Later we need two important theorems, which characterise compactness of subsets in Hölder- and L^p-spaces. The first one is the theorem of Arzela-Ascoli. We follow [3].

3.1.17 Theorem (Arzela-Ascoli)**.** *Let $n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ open. Then the closure of a set $\Lambda \subset C^0(\bar{\Omega})$ is compact if and only if for every $x \in \bar{\Omega}$ the set*

$$\Lambda(x) = \{f(x) \colon f \in \Lambda\}$$

is bounded and Λ is equicontinuous, i.e.

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall f \in \Lambda \ \forall x, y \in \bar{\Omega} \colon \quad |x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \epsilon.$$

Proof. "$\Rightarrow$": Compact sets in metric spaces are always bounded, since they can be covered by finitely many balls. Hence $\bar{\Lambda} \subset C^0(\bar{\Omega})$ is bounded and hence for all $x \in \bar{\Omega}$ and $f \in \Lambda$:

$$|f(x)| \leq |f|_{0,\Omega} \leq c.$$

If Λ was not equicontinuous, then there existed $\epsilon > 0$ and sequences $(f_n)_{n\in\mathbb{N}}$ in Λ and $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in $\bar{\Omega}$, such that

$$|x_n - y_n| < \frac{1}{n}, \quad |f_n(x_n) - f_n(y_n)| \geq \epsilon.$$

Due to the compactness of $\bar{\Lambda}$ and $\bar{\Omega}$, we find a sequence of indices n_k such that

$$f_{n_k} \to f \in C^0(\bar{\Omega}), \quad x_{n_k} \to x, \quad y_{n_k} \to x.$$

But then

$$\begin{aligned} |f_{n_k}(x_{n_k}) - f_{n_k}(y_{n_k})| &\leq |f_{n_k}(x_{n_k}) - f(x_{n_k})| + |f(x_{n_k}) - f(y_{n_k})| \\ &\quad + |f(y_{n_k}) - f_{n_k}(y_{n_k})| \\ &\leq 2|f_{n_k} - f|_{0,\Omega} + |f(x_{n_k}) - f(y_{n_k})| \\ &\to 0, \end{aligned}$$

a contradiction.

"$\Leftarrow$": Ω has a countable dense subset $D = \{y_k\}_{k\in\mathbb{N}}$, e.g. $\mathbb{Q}^n \cap \bar{\Omega}$. We have to prove that every sequence $(f_n)_{n\in\mathbb{N}}$ in $\bar{\Lambda}$ has a uniformly convergent subsequence. Setting

$$\begin{aligned} g \colon \bar{\Lambda} \times \bar{\Omega} &\to \mathbb{R} \\ g(f, y) &= f(y), \end{aligned}$$

we see that g, (f_n) and (y_k) satisfy the assumption of Cantor's diagonal sequence lemma, Lemma 3.1.10, and hence there exists a subsequence $(f_i)_{i\in\mathbb{N}}$ that converges pointwise,

$$\forall k \in \mathbb{N} \ \exists \alpha_k \in \mathbb{R} \colon \lim_{i\to\infty} f_i(y_k) = \alpha_k =: f(y_k).$$

The function $f \colon D \to \mathbb{R}$ is uniformly continuous, since for $\epsilon > 0$ we may pick $\delta > 0$ such that for all i there holds

$$|y_k - y_m| < \delta \quad \Rightarrow \quad |f_i(y_k) - f_i(y_m)| < \epsilon$$

and hence for $|y_k - y_m| < \delta$ there holds

$$\begin{aligned}|f(y_k) - f(y_m)| &\le |f(y_k) - f_i(y_k)| + |f_i(y_k) - f_i(y_m)| + |f_i(y_m) - f(y_m)| \\ &\le |f(y_k) - f_i(y_k)| + \epsilon + |f_i(y_m) - f(y_m)|\end{aligned}$$

and

$$|f(y_k) - f(y_m)| = \limsup_{i\to\infty} |f(y_k) - f(y_m)| \le \epsilon.$$

Thus f is uniformly continuous on a dense subset of $\bar{\Omega}$ and may be extended uniquely to a continuous function on $\bar{\Omega}$.[4] All that is left to show is the uniform convergence of f_i to f. Let $\epsilon > 0$ and pick $\delta > 0$ such that for all $h \in \{f_i\}_{i\in\mathbb{N}} \cup \{f\}$ there holds

$$|x - y| < \delta \quad\Rightarrow\quad |h(x) - h(y)| < \epsilon.$$

Then finitely many of the balls $B_\delta(y_k)$ cover $\bar{\Omega}$ and we obtain for all $x \in \bar{\Omega}$:

$$\begin{aligned}|f(x) - f_i(x)| &\le \min_{1\le j\le N} \big(|f(x) - f(y_{k_j})| + |f_i(y_{k_j}) - f_i(x)|\big) \\ &\quad + \max_{1\le j\le N} |f(y_{k_j}) - f_i(y_{k_j})| \\ &\le 2\epsilon + \max_{1\le j\le N} |f(y_{k_j}) - f_i(y_{k_j})|\end{aligned}$$

and the uniform convergence follows. □

3.1.18 Corollary. *Let $n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ open and $0 < \alpha \le 1$. Then the inclusion map*

$$C^{0,\alpha}(\bar{\Omega}) \hookrightarrow C^0(\bar{\Omega})$$

is compact.

Proof. First of all, every function $f \in C^{0,\alpha}(\bar{\Omega})$ extends to a continuous function on $\bar{\Omega}$. Let $(f_n)_{n\in\mathbb{N}}$ be bounded in $C^{0,\alpha}(\bar{\Omega})$, then

$$|f_n(x) - f_n(y)| \le [f_n]_{\alpha,\Omega}|x - y| \le c|x - y|$$

and hence the set $\{f_n\}_{n\in\mathbb{N}}$ is pointwisely bounded and equicontinuous. By Arzela-Ascoli it has a uniformly convergent subsequence. □

A similar result holds in L^p-spaces.

3.1.19 Theorem (Kolmogorov)**.** *Let $n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ open and $1 \le p < \infty$. The closure of a subset $M \subset L^p(\Omega)$ is compact if and only if M[5] is bounded and equicontinuous in the mean, i.e.*

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall u \in M\colon \quad 0 \le |h| < \delta \quad\Rightarrow\quad \|u - u(\cdot + h)\|_{p,\mathbb{R}^n} < \epsilon.$$

Proof. Let $\bar{M} \subset L^p(\Omega)$ be compact. Then M is bounded. Let $\epsilon > 0$. Then there exist $(u_i)_{1\le i\le N}$ in $\bar{M}$, such that

$$M \subset \bigcup_{i=1}^N B_\epsilon(u_i).$$

[4] This statement shall be proven as an exercise.

[5] More precisely: The set $\tilde{M}$ of functions in M, which are extended to $\mathbb{R}^n$ by zero

Let $u \in M$, then $u \in B_\epsilon(u_{i_0})$ and

$$\begin{aligned}\|u(\cdot+h)-u\|_{p,\mathbb{R}^n} &\leq \min_{1\leq i\leq N}(\|u(\cdot+h)-u_i(\cdot+h)\|_{p,\mathbb{R}^n}+\|u_i-u\|_{p,\mathbb{R}^n})\\ &\quad+\max_{1\leq i\leq N}\|u_i(\cdot+h)-u_i\|_{p,\mathbb{R}^n}\\ &< 2\epsilon+\max_{1\leq i\leq N}\|u_i(\cdot+h)-u_i\|_{p,\mathbb{R}^n},\end{aligned}$$

and the equicontinuity in the mean follows. We have used that a finite collection of functions is equicontinuous in the mean.[6]

Now suppose M is bounded and equicontinuous in the mean. Then this is also true for $\bar{M}$. For $\delta > 0$ let (η_δ) be a Dirac sequence. Let

$$u_\delta = u * \eta_\delta.$$

Then

$$\begin{aligned}|u_\delta(x)-u(x)|^p &= \left|\int_{B_\delta(0)}\eta_\delta(y)(u(x-y)-u(x))\right|^p dy\\ &= \left|\int_{B_\delta(0)}\eta_\delta^{\frac{p-1}{p}}(y)\eta_\delta^{\frac{1}{p}}(y)(u(x-y)-u(x))\right|^p dy\\ &\leq \int_{B_\delta(0)}\eta_\delta(y)|u(x-y)-u(x)|^p dy\end{aligned}$$

and hence

$$\int_{\mathbb{R}^n}|u_\delta-u|^p \leq \int_{B_\delta(0)}\eta_\delta(y)\int_{\mathbb{R}^n}|u(x-y)-u(x)|^p dxdy.$$

The equicontinuity implies

$$\sup_{u\in\bar{M}}\|u_\delta-u\|_{p,\mathbb{R}^n} \leq \sup_{u\in\bar{M}}\sup_{|y|<\delta}\|u(x-y)-u(x)\|_{p,\mathbb{R}^n}\to 0 \tag{3.4}$$

as $\delta \to 0$.

Now we claim that the closure of $M_\delta := \{u_\delta : u \in \bar{M}\} \subset C^0(\bar{\Omega}) =: E$[7] is compact in E. We have for any $x \in \bar{\Omega}$ and $u \in \bar{M}$:

$$\begin{aligned}|u_\delta(x)| &\leq \int_{B_\delta(0)}\eta_\delta^{1-\frac{1}{p}}(y)\eta_\delta^{\frac{1}{p}}(y)|u(x-y)|dy\\ &\leq \left(\int_{B_\delta(0)}\eta_\delta(y)|u(x-y)|^p\, dy\right)^{\frac{1}{p}}\\ &\leq \sup_{B_\delta}|\eta_\delta|^{\frac{1}{p}}\|u\|_{p,\mathbb{R}^n}\leq c.\end{aligned}$$

Furthermore, for all $x \in \Omega$ and $x+h \in \Omega$,

$$\begin{aligned}|u_\delta(x+h)-u_\delta(x)| &\leq \int_{B_\delta(0)}\eta_\delta^{1-\frac{1}{p}}(y)\eta_\delta^{\frac{1}{p}}(y)|u(x+h-y)-u(x-y)|dy\\ &\leq \sup_{B_\delta(0)}|\eta_\delta|^{\frac{1}{p}}\|u(\cdot+h)-u\|_{p,\mathbb{R}^n}.\end{aligned}$$

[6] The proof of this is an exercise.
[7] We restrict every u_δ to $\bar{\Omega}$.

Thus M_δ is equicontinuous and by Arzela-Ascoli $\bar{M}_\delta$ is compact in $C^0(\bar{\Omega})$. Now let $(u^n)_{n\in\mathbb{N}}$ be a sequence in $\bar{M}$ and $(\delta_k)_{k\in\mathbb{N}}$ a sequence with $\delta_k \to 0$. The map

$$\begin{aligned} g\colon \bar{M} \times (0,\infty) &\to C^0(\bar{\Omega}) \\ (u,\delta) &\mapsto u_\delta \end{aligned}$$

and the sequences $(u^n)_{n\in\mathbb{N}}$ and $(\delta_k)_{k\in\mathbb{N}}$ satisfy the assumption of Cantor's diagonal lemma and hence there is a subsequence $(u^i)_{i\in\mathbb{N}}$ such that for every $k \in \mathbb{N}$

$$u^i_{\delta_k} \to v_{\delta_k}$$

in $C^0(\bar{\Omega})$. We claim that $(u^i_{\delta_i})_{i\in\mathbb{N}}$ is an $L^p(\Omega)$-Cauchy sequence:

$$\begin{aligned} \|u^i_{\delta_i} - u^j_{\delta_j}\|_{p,\Omega} &\le \|u^i_{\delta_i} - u^i_{\delta_k}\|_{p,\Omega} + \|u^i_{\delta_k} - u^j_{\delta_k}\|_{p,\Omega} + \|u^j_{\delta_k} - u^j_{\delta_j}\|_{p,\Omega} \\ &\le \sup_{u\in\bar{M}} \|u_{\delta_i} - u_{\delta_k}\|_{p,\Omega} + \sup_{u\in\bar{M}} \|u_{\delta_k} - u_{\delta_j}\|_{p,\Omega} + \|u^i_{\delta_k} - u^j_{\delta_k}\|_{p,\Omega} \\ &\le \sup_{u\in\bar{M}} \|u_{\delta_i} - u\|_{p,\Omega} + \sup_{u\in\bar{M}} \|u - u_{\delta_j}\|_{p,\Omega} + 2\sup_{u\in\bar{M}} \|u - u_{\delta_k}\|_{p,\Omega} \\ &\quad + \|u^i_{\delta_k} - u^j_{\delta_k}\|_{p,\Omega}. \end{aligned}$$

Due to (3.4) we may, for given $\epsilon > 0$, pick k so large that

$$\|u^i_{\delta_i} - u^j_{\delta_j}\|_{p,\Omega} \le 2\epsilon + \sup_{u\in\bar{M}} \|u_{\delta_i} - u\|_{p,\Omega} + \sup_{u\in\bar{M}} \|u - u_{\delta_j}\|_{p,\Omega} + \|u^i_{\delta_k} - u^j_{\delta_k}\|_{p,\Omega}.$$

Picking i, j large enough gives

$$\|u^i_{\delta_i} - u^j_{\delta_j}\|_{p,\Omega} \le 5\epsilon.$$

Hence there exists $v \in L^p(\Omega)$ such that

$$u^i_{\delta_i} \to v.$$

Furthermore

$$\|u^i - u^i_{\delta_i}\|_{p,\Omega} \le \sup_{u\in\bar{M}} \|u - u_{\delta_i}\|_{p,\Omega} \to 0$$

as $i \to \infty$ and hence (u^i) is the desired convergent subsequence. □

Abstract eigenvalue problems

Due the maximum principle we know that any solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ of

$$\begin{aligned} -\Delta u &= \lambda u \text{ in } \Omega \\ u_{|\partial\Omega} &= 0 \end{aligned}$$

must be zero, provided $\lambda \le 0$. What happens in case $\lambda > 0$? Put in a different way: Are there nontrivial eigenfunctions of the Laplace operator with Dirichlet boundary condition? In this subsection we will answer this question in an abstract Hilbert space setting to prove that there are weak eigenfunctions. Later we will see that these are actually smooth.

3.1.20 Lemma. *Let (H, g) be a real Hilbert space, K a symmetric, continuous and compact bilinear form*[8] *on H, such that*

$$\forall u \neq 0\colon K(u) := K(u,u) > 0$$

and B a symmetric, continuous bilinear form on H, which is coercive relative K, i.e.

$$\exists c_0, c > 0 \ \forall u \in H\colon B(u) := B(u,u) \geq c\|u\|_g^2 - c_0 K(u).$$

Let $\{0\} \neq V \subset H$ be a closed subspace. Then the variational problem

$$B(v) \to \min, \ v \in W := V \cap \{K(v) = 1\}$$

has a solution u, which is also a solution of

$$\frac{B(v)}{K(v)} \to \min, \ 0 \neq v \in V.$$

Setting

$$\lambda = \inf_{0 \neq v \in V} \frac{B(v)}{K(v)},$$

then we have

$$\forall v \in V\colon B(u,v) = \lambda K(u,v).$$

Proof. By coercivity we see, that B is bounded below in W and that a minimal sequence u_ϵ is bounded above. Thus we suppose

$$u_\epsilon \rightharpoonup u \in V$$

and deduce

$$K(u_\epsilon) \to K(u) = 1.$$

Because $B + c_0 K$ is an equivalent norm on H, $B + c_0 K$ is weakly lower semi-continuous and hence

$$B(u) + c_0 K(u) \leq \liminf_{\epsilon \to 0} \left(B(u_\epsilon) + c_0 K(u_\epsilon) \right),$$

which implies

$$B(u) \leq \liminf_{\epsilon \to 0} B(u_\epsilon).$$

Thus the first two claims follow. We calculate the first variation of

$$v \mapsto \frac{B(v)}{K(v)}$$

at the minimum u:

$$0 = \frac{d}{dt} \frac{B(u+tv)}{K(u+tv)}_{|t=0} = \frac{2B(u,v)}{K(u)} - \frac{2B(u)K(u,v)}{K(u)^2}$$

and hence, for all $v \in V$,

$$B(u,v) = \frac{B(u)}{K(u)} K(u,v) = \lambda K(u,v).$$

□

[8] Every bounded sequence in H has a subsequence which converges in the norm induced by K.

3.1.21 Theorem. *Let (H, g) be an infinite dimensional real Hilbert space, K a symmetric, continuous and compact bilinear form on H, such that*

$$\forall u \neq 0\colon K(u) := K(u, u) > 0$$

and B a symmetric, continuous bilinear form on H, which is coercive relative K, i.e.

$$\exists c_0, c > 0 \ \forall u \in H\colon B(u) := B(u, u) \geq c\|u\|_g^2 - c_0 K(u).$$

Then the eigenvalue problem

$$\exists 0 \neq u_i \in H \ \forall v \in H\colon B(u_i, v) = \lambda_i K(u_i, v)$$

has countably many solutions λ_i of finite multiplicity. If we write

$$\lambda_1 \leq \lambda_2 \leq ...,$$

we obtain

$$\lim_{i \to \infty} \lambda_i = \infty.$$

The eigenvectors (u_i) are complete in H.[9] *They satisfy the orthogonality relations*

$$K(u_i, u_j) = \delta_{ij}$$

and

$$B(u_i, u_j) = \lambda_i K(u_i, u_j),$$

as well as the expansions

$$B(u, v) = \sum_i \lambda_i K(u_i, u) K(u_i, v)$$

and

$$K(u, v) = \sum_i K(u_i, u) K(u_i, v).$$

The pairs (λ_i, u_i) are defined by the variational problem

$$\lambda_i = B(u_i, u_i) = \inf \left\{ \frac{B(u)}{K(u)} \colon 0 \neq u \in H, K(u, u_j) = 0 \ \forall 1 \leq j \leq i-1 \right\}.$$

Proof. Step 1: Solve the variational problem

$$\frac{B(u)}{K(u)} \to \min, \ 0 \neq u \in H.$$

By the previous theorem there exists a solution u_1 and there holds

$$\forall v \in H\colon B(u_1, v) = \lambda_1 K(u_1, v), \ K(u_1) = 1,$$

such that λ_1 is the infimum.
Step 2: Let $i > 1$ and let there be solutions for $1 \leq j \leq i-1$. Set

$$V_i = \operatorname{span}(u_1, ..., u_{i-1})$$

[9] $\operatorname{span}(u_i)$ is dense in H.

and let $V_i^\perp$ be the orthogonal complement of V relative K. Again, by the previous theorem

$$\exists u_i \in V_i^\perp : B(u_i) = \lambda_i = \inf \left\{ \frac{B(u)}{K(u)} : u \in V_i^\perp \right\}$$

and

$$\forall v \in V^\perp : B(u_i, v) = \lambda_i K(u_i, v).$$

For $1 \leq j \leq i-1$ we have

$$B(u_j, u_i) = \lambda_j K(u_j, u_i) = 0.$$

Thus

$$\forall v \in H : B(u_i, v) = \lambda_i K(u_i, v),$$

since

$$H = V_i \oplus_K V_i^\perp.$$

The u_i satisfy the orthogonality relation

$$B(u_i, u_j) = \lambda_i K(u_i, u_j) = \lambda_i \delta_{ij}.$$

Step 3: Suppose now the eigenvalues were bounded. We have

$$B(u_i) = \lambda_i, \quad K(u_i) = 1,$$

and thus

$$c_0 K(u_i) + B(u_i) = \lambda_i + c_0,$$

so that the u_i are bounded. Hence

$$2 = K(u_i - u_{i+1}) \to 0$$

for a subsequence, which is a contradiction. By the same reasoning the multiplicity must be finite.
Step 4: We prove the completeness. Let $u \in H$. Define

$$\tilde{u}_m = \sum_{i=1}^m K(u, u_i) u_i \equiv \sum_{i=1}^m c_i u_i$$

and

$$v_m = u - \tilde{u}_m \in V_{m+1}^\perp.$$

Thus

$$\lambda_{m+1} K(v_m) \leq B(v_m)$$

and

$$K(v_m) = K(u) - \sum_{i=1}^m c_i^2, \quad B(v_m) = B(u) - \sum_{i=1}^m \lambda_i c_i^2$$

imply

$$B(v_m) \leq c$$

and thus

$$K(v_m) \to 0.$$

Furthermore there holds

$$\sum_{i=1}^{\infty} \lambda_i c_i^2 < \infty.$$

Let $m < n$.

$$B(v_n - v_m) = \sum_{i=m+1}^{n} \lambda_i c_i^2 \to 0.$$

Thus the (v_n) form a Cauchy sequence in H and by $K(v_m) \to 0$ we find

$$v_m \to 0.$$

This implies that the (u_i) are complete and

$$B(u) = \sum_{i=1}^{\infty} \lambda_i c_i^2.$$

□

3.2 Distributions

The theory of distributions ensures the possibility to define derivatives of very general objects, which will suffice for all our purposes. We will restrict to the very basics here.

3.2.1 Definition (Test functions). Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ open.

(i) The elements of $C_c^\infty(\Omega)$ are called *test functions*.

(ii) We define a sequence $(\varphi_k)_{k\in\mathbb{N}}$ in $C_c^\infty(\Omega)$ to converge to $\varphi \in C_c^\infty(\Omega)$, if

   (a) $\exists k_0 \in \mathbb{N}\ \exists \Omega' \Subset \Omega\ \forall k \geq k_0 \colon \operatorname{supp} \varphi_k \subset \Omega'$ and

   (b) $|\varphi_k - \varphi|_{m,\Omega'} \to 0 \quad \forall m \in \mathbb{N}$.

(iii) The vector space $C_c^\infty(\Omega)$ equipped with this notion of convergence is denoted by $\mathcal{D}(\Omega)$.

3.2.2 Definition (Distributions). Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ open.

(i) A *distribution on* Ω is a continuous linear map

$$\Theta \colon \mathcal{D}(\Omega) \to \mathbb{R}.$$

We write $\mathcal{D}'(\Omega)$ for the set of distributions on Ω.

(ii) A sequence $(\Theta_k)_{k\in\mathbb{N}}$ of distributions is said to converge to a distribution Θ, if it converges pointwise,

$$\forall \varphi \in \mathcal{D}(\Omega) \colon \Theta_k(\varphi) \to \Theta(\varphi).$$

3.2.3 Example. (i) Let $f \in L^1_{\text{loc}}(\Omega)$, then

$$\Theta_f(\varphi) = \int_\Omega f\varphi$$

defines a distribution and the assignment

$$f \mapsto \Theta_f$$

is injective, as you can check as an easy exercise.

(ii) For $x \in \Omega$ the map

$$\begin{aligned} \delta_x \colon \mathcal{D}(\Omega) &\to \mathbb{R} \\ \varphi &\mapsto \varphi(x) \end{aligned}$$

is a distribution, the so-called *Dirac-delta distribution.*

(iii) Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a mollifier, then the family

$$\eta_\epsilon(x) = \epsilon^{-n}\eta\left(\frac{x}{\epsilon}\right)$$

is also called the *Dirac sequence of* η. The reason for this is apparent from the property that for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ there holds

$$\Theta_{\eta_\epsilon(x-\cdot)}(\varphi) = \int_{\mathbb{R}^n} \eta_\epsilon(x-y)\varphi(y)\,dy \to \varphi(x) = \delta_x(\varphi).$$

Hence

$$\Theta_{\eta_\epsilon(x-\cdot)} \to \delta_x, \quad \epsilon \to 0.$$

Now we define the derivative of a distribution. This definition is motivated from the rule of partial integration.

3.2.4 Definition (Distributional derivative)**.** Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open, $\Theta \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^n$.

(i) We define the *α-th distributional derivative* of Θ, $\Theta_{,\alpha} \in \mathcal{D}'(\Omega)$, by

$$\Theta_{,\alpha}(\varphi) := (-1)^{\langle\alpha\rangle}\Theta(\varphi_{,\alpha}). \tag{3.5}$$

(ii) In case that Θ arises from a function $f \in L^1_{\text{loc}}(\Omega)$ as in Example 3.2.3, we write

$$f_{,\alpha} := (\Theta_f)_{,\alpha}$$

and call $f_{,\alpha}$ the *α-th weak derivative* of f.

3.2.5 Remark. This definition is cooked up, such that it really is a generalization of ordinary differentiation and at the same time a certain rule of partial integration holds, namely (3.5). It generalizes differentiation, since for $f \in C^{\langle\alpha\rangle}(\Omega) \subset L^1_{\text{loc}}(\Omega)$ there holds

$$(\Theta_f)_{,\alpha}(\varphi) = (-1)^{\langle\alpha\rangle}\int_\Omega f\varphi_{,\alpha} \overset{10}{=} \int_\Omega f_{,\alpha}\varphi = \Theta_{f_{,\alpha}}(\varphi).$$

In this sense the distributional derivative of f coincides with the classical derivative.

[10] Classical partial integration.

3.2.6 Example (Heavyside function). Let $\vartheta \in L^1_{\mathrm{loc}}(\mathbb{R})$ be given by

$$\vartheta(t) := \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$$

Then, as one may verify as an exercise, $\vartheta' = 2\delta_0$, where δ_0 is the Dirac-delta distribution in $0 \in \mathbb{R}$.

3.3 Sobolev spaces

Definition and smooth approximation

3.3.1 Definition. Let $n, m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open and $1 \leq p \leq \infty$.

(i) Define the *Sobolev space of class* (m,p) by

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) \colon u_{,\alpha} \in L^p(\Omega)\ \forall \langle\alpha\rangle \leq m\}$$

and equip it with the norm

$$\|u\|_{m,p,\Omega} = \left(\sum_{\langle\alpha\rangle \leq m} \|u_{,\alpha}\|^p_{p,\Omega} \right)^{\frac{1}{p}}$$

in case $1 \leq p < \infty$ and

$$\|u\|_{m,\infty,\Omega} = \max_{\langle\alpha\rangle \leq m} \|u_{,\alpha}\|_{\infty,\Omega}$$

in case $p = \infty$. On $W^{m,2}(\Omega)$ we define the scalar product

$$\langle u, v\rangle_{m,2,\Omega} := \sum_{\langle\alpha\rangle \leq m} \int_\Omega u_{,\alpha} v_{,\alpha}.$$

(ii) Functions u belonging to $W^{1,1}_{\mathrm{loc}}(\Omega)$ are called *weakly differentiable* and their distributional derivatives are called the *weak derivatives.*

(iii) We define the *local Sobolev space of class* (m,p) by

$$W^{m,p}_{\mathrm{loc}}(\Omega) = \{u \in L^p_{\mathrm{loc}}(\Omega) \colon u \in W^{m,p}(\Omega') \quad \forall \Omega' \Subset \Omega\}.$$

(iv) We define $W^{m,p}_0(\Omega)$ to be the closure of $C^\infty_c(\Omega)$ with respect to $\|\cdot\|_{m,p,\Omega}$.

This indeed is a generalisation of classical differentiation:

3.3.2 Exercise. Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ and $1 \leq p \leq \infty$. Then

$$C^m(\Omega) \subset W^{m,p}_{\mathrm{loc}}(\Omega)$$

and for $\langle\alpha\rangle \leq m$ the α-th weak derivative of a function $u \in C^m(\Omega)$ can be represented by[11] by the classical α-partial derivative $\partial_\alpha u$.

[11] i.e. is up to measure zero given by

3.3.3 Proposition. *Let $n, m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open and $1 \le p \le \infty$. Then $W^{m,p}(\Omega)$ is a Banach space and for $p = 2$ it is a Hilbert space.*

Proof. Let $(u_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $W^{m,p}(\Omega)$. In particular it is a Cauchy sequence in $L^p(\Omega)$ and hence has a limit $u \in L^p(\Omega)$. Furthermore for all α with $\langle\alpha\rangle \le m$ the sequence $(u_{k,\alpha})$ is a Cauchy sequence in $L^p(\Omega)$ and hence converges to some limit $g_{,\alpha} \in L^p(\Omega)$. There holds for alle test functions $\varphi \in C_c^\infty(\Omega)$, that

$$\int_\Omega g_{,\alpha}\varphi = \lim_{k\to\infty} \int_\Omega u_{k,\alpha}\varphi = (-1)^{\langle\alpha\rangle} \lim_{k\to\infty} \int_\Omega u_k \varphi_{,\alpha} = (-1)^{\langle\alpha\rangle} \int_\Omega u\varphi_{,\alpha}.$$

Hence we have calculated the α-th distributional derivative of u to be $g_{,\alpha} \in L^p(\Omega)$ and hence

$$u \in W^{m,p}(\Omega)$$

and $u_k \to u$ in $W^{m,p}(\Omega)$.

$W^{m,2}(\Omega)$ is a Hilbert space because the inner product $\langle\cdot,\cdot\rangle_{m,2,\Omega}$ induces the Sobolev norm. □

Also for Sobolev functions we obtain a smoothing result, the proof of which is an exercise.

3.3.4 Exercise. Let $n, m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open, $1 \le p < \infty$, η a mollifier with Dirac sequence (η_ϵ), $u \in W^{m,p}(\Omega)$ and u_ϵ their convolutions. Then for all $\Omega' \subset \Omega$ with

$$\bar{\Omega}' \subset \Omega, \quad \operatorname{dist}(\bar{\Omega}', \partial\Omega) > 0$$

there holds

$$\lim_{\epsilon\to 0} \|u_\epsilon - u\|_{m,p,\Omega'} = 0.$$

Basic rules for calculation

Many properties of classically differentiable functions carry over to Sobolev functions due to these approximation properties. We prove some of them now.

3.3.5 Proposition (Product rule)**.** *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open, $1 \le p, p' \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let $u \in W^{1,p}(\Omega)$ and $v \in W^{1,p'}(\Omega)$. Then*

$$uv \in W^{1,1}(\Omega)$$

and

$$D(uv) = Du \cdot v + u \cdot Dv.$$

Proof. By symmetry we may assume $p < \infty$. Let $\varphi \in C_c^\infty(\Omega)$ with

$$\operatorname{supp}\varphi \subset \Omega' \Subset \Omega$$

and let $\epsilon < \operatorname{dist}(\Omega', \partial\Omega)$. Let u_ϵ be the convolution of u with a Dirac sequence. Then there holds

$$\int_{\Omega'} (\varphi u_\epsilon) v_{,i} = -\int_{\Omega'} (\varphi u_{\epsilon,i} v + u_\epsilon \varphi_{,i} v).$$

Taking the limit $\epsilon \to 0$ and reverting Ω' back to Ω we obtain

$$\forall \varphi \in C_c^\infty(\Omega)\colon \int_\Omega \varphi(uv_{,i} + u_{,i}v) = -\int_\Omega uv\varphi_{,i}.$$

This proves the product rule and from Hölder's inequality we get

$$D(uv) \in L^1(\Omega).$$

□

3.3.6 Proposition (Chain rule)**.** *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open, $1 \le p \le \infty$ and $g \in C^1(\mathbb{R})$ with bounded derivative. Let $u \in W^{1,p}(\Omega)$. Then, if $g \circ u \in L^p(\Omega)$, we have $g \circ u \in W^{1,p}(\Omega)$ and*

$$D(g \circ u) = g'(u)Du.$$

Proof. Let $\varphi \in C_c^\infty(\Omega)$ and $\Omega' \Subset \Omega$, such that $\varphi \in C_c^\infty(\Omega')$. Let $u_\epsilon \in C^\infty(\Omega')$ be the convolution with a Dirac sequence η_ϵ, such that

$$\|u - u_\epsilon\|_{1,1,\Omega'} \to 0$$

and

$$(u_\epsilon, Du_\epsilon) \to (u, Du) \text{ a.e.}$$

Then $g \circ u_\epsilon \to g \circ u$ in $L^1(\Omega')$, since g is uniformly Lipschitz continuous and hence

$$\int_{\Omega'} (g \circ u)\varphi_{,i} = \lim_{\epsilon \to 0} \int_{\Omega'} (g \circ u_\epsilon)\varphi_{,i} = \lim_{\epsilon \to 0} \left(-\int_{\Omega'} g'(u_\epsilon)u_{\epsilon,i}\varphi \right) \tag{3.6}$$

There holds $g'(u_\epsilon) \to g'(u)$ a.e. and $|g'| \le L$. Hence

$$|\varphi g'(u_\epsilon)Du| \le L|Du||\varphi|.$$

Dominated convergence implies

$$\begin{aligned} \int_{\Omega'} |g'(u_\epsilon)u_{\epsilon,i}\varphi - g'(u)u_{,i}\varphi| &\le \int_{\Omega'} |g'(u_\epsilon)(u_{\epsilon,i} - u_{,i})\varphi| \\ &\quad + \int_{\Omega'} |g'(u_\epsilon) - g'(u)||Du||\varphi| \to 0. \end{aligned}$$

(3.6) implies the chain rule. $g \circ u \in W^{1,p}(\Omega)$ follows immediately. □

3.3.7 Proposition. *Let $n, m \in \mathbb{N}$, $\Omega, \tilde{\Omega} \subset \mathbb{R}^n$ open and $1 \le p \le \infty$. Let $u \in W^{m,p}(\Omega)$ and $\psi = (\tilde{x}^i) \in C^m(\Omega, \tilde{\Omega})$ be a coordinate transformation, such that ψ and ψ^{-1} have a bounded derivatives up to order m. Then the map $\tilde{u} = u \circ \psi^{-1}$ belongs to $W^{m,p}(\tilde{\Omega})$ and there holds*

$$\tilde{u}_{,i} = (u_{,k} \circ \psi^{-1})x^k_{,i}, \tag{3.7}$$

where (x^k) denote the component functions of ψ^{-1}, i.e. $\psi^{-1}(\tilde{x}) = (x^k(\tilde{x}))$. Furthermore there holds

$$\|\tilde{u}\|_{m,p,\tilde{\Omega}} \le c\|u\|_{m,p,\Omega}.$$

Proof. First suppose $m = 1$. Let $\varphi \in C_c^\infty(\tilde{\Omega})$ and $\Omega' \Subset \Omega$, such that $\varphi \in C_c^\infty(\psi(\Omega'))$ and $u_\epsilon \to u$ in $W^{1,1}(\Omega')$ an approximation by convolutions with a Dirac sequence. Define

$$\tilde{u}_\epsilon = u_\epsilon \circ \psi^{-1}.$$

Then

$$\tilde{u}_{\epsilon,i} = u_{\epsilon,k} x^k_{,i}.$$

Due to the transformation theorem we have $\tilde{u}_\epsilon \to \tilde{u}$ in $L^1(\psi(\Omega'))$ and

$$\tilde{u}_{\epsilon,i} \to u_{,k} x^k_{,i}$$

in $L^1(\psi(\Omega'))$. Hence

$$\int_{\psi(\Omega')} \tilde{u}\varphi_{,i} = \lim_{\epsilon\to 0} \int_{\psi(\Omega')} \tilde{u}_\epsilon \varphi_{,i} = -\lim_{\epsilon\to 0} \int_{\psi(\Omega')} u_{\epsilon,k} x^k_{,i}\varphi = -\int_{\psi(\Omega')} u_{,k} x^k_{,i}\varphi.$$

By the transformation theorem and the boundedness of the Jacobians we obtain

$$\|\tilde{u}\|_{1,p,\tilde{\Omega}} \le c\|u\|_{1,p,\Omega}$$

in case $p < \infty$, while in case $p = \infty$ this estimate is trivial. For $m > 1$ we proceed by induction. Let the result be valid for $m \ge 1$, then by (3.7) we obtain that

$$\tilde{u}_{,i}\tilde{x}^i_{,k} \in W^{m-1,p}(\Omega)$$

with the estimate

$$\|\tilde{u}_{,i}\tilde{x}^i_{,k}\|_{m-1,p,\tilde{\Omega}} \le c\|u_{,k}\|_{m-1,p,\Omega} \le c\|u\|_{m,p,\Omega}.$$

Due to

$$\tilde{u}_{,i} = \tilde{u}_{,k}\tilde{x}^k_{,m} x^m_{,i},$$

an inductive use of the product rule and the boundedness of all derivatives of ψ, we obtain the desired estimate. □

3.3.8 Lemma. *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open, $1 \le p \le \infty$ and $u \in W^{1,p}(\Omega)$. Then*

$$u^+ = \max(u, 0), \quad u^- = \min(u, 0)$$

are in $W^{1,p}(\Omega)$ and there holds

$$Du^+ = \begin{cases} Du, & u > 0 \\ 0, & u \le 0 \end{cases},$$

$$Du^- = \begin{cases} Du, & u < 0 \\ 0, & u \ge 0 \end{cases}.$$

Proof. Let $\epsilon > 0$ and set

$$g_\epsilon(t) := \begin{cases} \sqrt{t^2 + \epsilon^2} - \epsilon, & t > 0 \\ 0, & t \le 0. \end{cases}$$

Then $g_\epsilon \in C^1(\mathbb{R})$ and $|g'_\epsilon| \leq 1$. Since $g_\epsilon(0) = 0$, we have $g_\epsilon \circ u \in L^p(\Omega)$ and hence the chain rule implies

$$u_\epsilon := g_\epsilon \circ u \in W^{1,p}(\Omega)$$

and

$$Du_\epsilon = g'_\epsilon(u)Du = \begin{cases} \frac{uDu}{\sqrt{u^2+\epsilon^2}}, & u > 0 \\ 0, & u \leq 0. \end{cases}$$

Let $\varphi \in C_c^\infty(\Omega)$. Then, due to $0 \leq u_\epsilon \leq u$ and the dominated convergence theorem,

$$\begin{aligned} \int_\Omega u^+ \varphi_{,i} = \lim_{\epsilon \to 0} \int_\Omega u_\epsilon \varphi_{,i} &= -\lim_{\epsilon \to 0} \int_\Omega u_{\epsilon,i} \varphi \\ &= -\lim_{\epsilon \to 0} \int_{\{u>0\}} \frac{uu_{,i}}{\sqrt{u^2+\epsilon^2}} \varphi \\ &= -\lim_{\epsilon \to 0} \int_\Omega \frac{uu_{,i}}{\sqrt{u^2+\epsilon^2}} \chi_{\{u>0\}} \varphi \\ &= -\int_\Omega \chi_{\{u>0\}} u_{,i} \varphi. \end{aligned}$$

Using $u^- = -(-u)^+$ the result for u^- follows. □

3.3.9 Exercise. Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open, $1 \leq p \leq \infty$ and $u \in W^{1,p}(\Omega)$. Then for all $c \in \mathbb{R}$ there holds

$$Du_{|\{u=c\}} = 0$$

almost everywhere.

Theorem of Meyers-Serrin

The main feature of the previous proofs is that we have always approximated Sobolev functions locally by smooth functions. Historically, an important step in the theory of Sobolev spaces was that actually

$$C^\infty(\Omega) \cap W^{m,p}(\Omega) \subset W^{m,p}(\Omega)$$

is dense.[12]

3.3.10 Theorem (Meyers-Serrin). *Let $n, m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open and $1 \leq p < \infty$. Then for all $u \in W^{m,p}(\Omega)$ there exists a sequence $(u_k)_{k \in \mathbb{N}}$ in $C^\infty(\Omega) \cap W^{m,p}(\Omega)$, such that*

$$\|u_k - u\|_{m,p,\Omega} \to 0.$$

[12] Before the 1960's it was also common to define Sobolev spaces as the completion of

$$\{u \in C^\infty(\Omega) \colon \|u\|_{m,p,\Omega} < \infty\}$$

under the $\|\cdot\|_{m,p,\Omega}$ norm. Those spaces were then called $H^{m,p}(\Omega)$. Since obviously there holds $H^{m,p}(\Omega) \subset W^{m,p}(\Omega)$, this theorem shows that $H^{m,p}(\Omega) = W^{m,p}(\Omega)$. This result is due to Meyers and Serrin and appeared in the beautiful paper $H = W$, cf. [12].

Proof. Let $(\Omega_j)_{j\in\mathbb{N}}$ be an exhaustion of Ω, i.e.

$$\Omega_j \Subset \Omega_{j+1}, \quad \Omega = \bigcup_{j\in\mathbb{N}} \Omega_j.$$

Let $U_j = \Omega_{j+1}\backslash\bar{\Omega}_{j-1}$, where $\Omega_0 = \Omega_{-1} := \emptyset$. Let (φ_i) be a countable partition of unity for Ω according to Theorem 1.3.8. Let $u \in W^{m,p}(\Omega)$ and $\epsilon > 0$, then for each i there exists j_i such that

$$\operatorname{supp}(\varphi_i u) \subset U_{j_i}.$$

Let

$$h_i < \min(\operatorname{dist}(U_{j_i}, \partial\Omega), \operatorname{dist}(\operatorname{supp}(\varphi_i u), \partial U_{j_i})),$$

such that for the h_i-convolution of $\varphi_i u$ we get

$$\|(\varphi_i u)_{h_i} - \varphi_i u\|_{m,p,\Omega} = \|(\varphi_i u)_{h_i} - \varphi_i u\|_{m,p,U_{j_i}} < \frac{\epsilon}{2^i}.$$

Defining

$$v(x) = \sum_{i\in\mathbb{N}} (\varphi_i u)_{h_i}(x),$$

which is a fixed finite sum as long as x ranges in any given $\Omega' \Subset \Omega$, we see that $v \in C^\infty(\Omega) \cap W^{m,p}(\Omega)$. Furthermore

$$\|v - u\|_{m,p,\Omega} \le \sum_{i\in\mathbb{N}} \|(\varphi_i u)_{h_i} - \varphi_i u\|_{m,p,\Omega} < \epsilon.$$

Hence $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense and the proof complete. □

Difference quotients

Soon we will see, that the space in which we search a solution of

$$\Delta u = f$$

is $W^{1,2}(\Omega)$, for example when $f \in L^2(\Omega)$. Of course it is then natural to ask, whether the solution is actually $W^{2,2}(\Omega)$, since this would be expected from counting the orders of derivatives. Since there is no way to directly estimate the second weak derivative of u in $L^2(\Omega)$, simply because it is not known yet to exist, we will instead look at *difference quotients*. The crucial results concerning difference quotients will be deduced in the sequel.

3.3.11 Definition (Difference quotients)**.** Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open and $u \in \mathbb{R}^\Omega$. Let $\Omega' \Subset \Omega$ and $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$, then for $1 \le i \le n$ we define the *difference quotient of u with stepsize h in direction e_i*, $\Delta_h^i u \in \mathbb{R}^{\Omega'}$, by

$$\Delta_h^i u(x) = \frac{u(x + he_i) - u(x)}{h}.$$

3.3.12 Lemma. *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open and $1 \le p \le \infty$. For $\Omega' \Subset \Omega$, $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$ and $1 \le i \le n$,*

$$\Delta_h^i : L^p(\Omega) \to L^p(\Omega')$$

is a continuous linear operator and

$$\|\Delta_h^i u\|_{p,\Omega'} \le \frac{2}{|h|}\|u\|_{p,\Omega}.$$

For $u, v \in L^2(\Omega)$ *there also holds*

$$\langle \Delta_h^i u, v\rangle_{2,\Omega} = -\langle u, \Delta_{-h}^i v\rangle_{2,\Omega}$$

if v *has compact support in* Ω'.[13]

Proof. The first statement is obvious. In case $p = 2$, w.l.o.g. let $\text{supp}(v) \subset \Omega'$. Then

$$\begin{aligned}
\langle \Delta_h^i u, v\rangle_{2,\Omega} &= \int_{\Omega'} \frac{u(x+he_i) - u(x)}{h} v(x)\, dx \\
&= \frac{1}{h}\int_{\Omega'} u(x+he_i)v(x)\, dx - \frac{1}{h}\int_{\Omega'} u(x)v(x)\, dx \\
&= \frac{1}{h}\int_{\Omega'+he_i} u(y)v(y-he_i)\, dy - \frac{1}{h}\int_{\Omega'} u(y)v(y)\, dy \\
&= -\int_{\Omega} u(y)\frac{v(y-he_i) - v(y)}{(-h)}\, dy,
\end{aligned}$$

where in the last step we have used that $\text{supp}(v) \subset \Omega'$. □

The following lemmata are the crucial results for difference quotients in the context of Sobolev spaces.

3.3.13 Lemma. *Let* $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ *open,* $1 \le p < \infty$, $u \in W^{1,p}(\Omega)$, $\Omega' \Subset \Omega$, $0 < |h| < \text{dist}(\Omega', \partial\Omega)$ *and* $1 \le i \le n$. *Then*

$$\|\Delta_h^i u\|_{p,\Omega'} \le \|u_{,i}\|_{p,\Omega}$$

and

$$\lim_{h\to 0} \|u_{,i} - \Delta_h^i u\|_{p,\Omega'} = 0.$$

Proof. Without loss of generality assume $i = n$. We use the notation

$$\hat{x} = (x^1, \dots, x^{n-1}).$$

First suppose $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$. Let $x \in \Omega'$.

$$\Delta_h^n u(x) = \frac{1}{h}\int_{x_n}^{x_n+h} u_{,n}(\hat{x}, t)\, dt$$

and thus using Hölder's inequality,

$$\begin{aligned}
|\Delta_h^n u(x)|^p &\le |h|^{-p}\left|\int_{x_n}^{x_n+h} u_{,n}(\hat{x}, t)\, dt\right|^p \\
&\le |h|^{-p}|h|^{p-1}\int_{x_n}^{x_n+h} |u_{,n}(\hat{x}, t)|^p\, dt \\
&= |h|^{-1}\int_{x_n}^{x_n+h} |u_{,n}(\hat{x}, t)|^p\, dt.
\end{aligned}$$

[13] In this equality u and v are extended to $\mathbb{R}^n$ by zero.

Thus we have

$$\int_{\Omega'} |\Delta_h^n u(x)|^p dx \le |h|^{-1} \int_0^h \int_{\Omega'} |u_{,n}(\hat{x}, x^n + t)|^p \; dxdt \le \|u_{,n}\|_{p,\Omega}^p.$$

By Lemma 3.3.12 both sides are continuous with respect to $W^{1,p}(\Omega)$ convergence and hence the result holds for general $u \in W^{1,p}(\Omega)$.

For the second claim let $\epsilon > 0$. Choose $v \in C^1(\Omega) \cap W^{1,p}(\Omega)$ such that

$$\|v - u\|_{1,p,\Omega} < \frac{\epsilon}{2}.$$

Then

$$\|u_{,n} - \Delta_h^n u\|_{p,\Omega'} \le \|u_{,n} - v_{,n}\|_{p,\Omega'} + \|v_{,n} - \Delta_h^n v\|_{p,\Omega'} + \|\Delta_h^n (u - v)\|_{p,\Omega'}$$

and hence

$$\limsup_{h\to 0} \|u_{,n} - \Delta_h^n u\|_{p,\Omega'} \le \epsilon + \limsup_{h\to 0} \|v_{,n} - \Delta_h^n v\|_{p,\Omega'} = \epsilon,$$

since for C^1-functions the difference quotients convergence locally uniformly to the derivative, due to the estimate of the remainder in Taylor's formula. □

The next lemma is valid for $u \in L^p(\Omega)$ with $1 < p < \infty$, but we only prove it for $p = 2$ in order to keep the required knowledge from functional analysis at a minimum.

3.3.14 Lemma. *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open, $u \in L^2(\Omega)$, $\Omega' \Subset \Omega$, $0 < h_0 < \operatorname{dist}(\Omega', \partial\Omega)$ and $1 \le i \le n$. Suppose*

$$\forall 0 < |h| < h_0 \colon \|\Delta_h^i u\|_{2,\Omega'} \le c.$$

Then the weak derivative $u_{,i}$ exists and

$$\|u_{,i}\|_{2,\Omega'} \le c.$$

Proof. $L^2(\Omega')$ is a Hilbert space. Thus there exists a sequence h_k such that

$$\Delta_{h_k}^i u \rightharpoonup v \in L^2(\Omega')$$

and

$$\|v\|_{2,\Omega'} \le \liminf_{k\to\infty} \|\Delta_{h_k}^i u\|_{2,\Omega'} \le c.$$

Let $\varphi \in C_c^\infty(\Omega')$ and pick $\Omega'' \Subset \Omega'$, auch that $\varphi \in C_c^\infty(\Omega'')$. Then

$$\langle v, \varphi \rangle_{2,\Omega'} = \lim_{k\to\infty} \langle \Delta_{h_k}^i u, \varphi \rangle_{2,\Omega''} = -\lim_{k\to\infty} \langle u, \Delta_{-h_k}^i \varphi \rangle_{2,\Omega'} = -\langle u, \varphi_{,i} \rangle.$$

Thus $v = u_{,i}$. □

3.4 Embedding and compactness theorems

In order to show that a sufficiently regular weak solution is actually differentiable, we need so-called embedding theorems for Sobolev spaces. Among other statements they imply that

$$W_0^{m,p}(\Omega) \subset C^k(\bar{\Omega}),$$

where $k = k(m,p)$ and m is large enough. This will in turn yield classical solutions to our PDE.

3.4.1 Theorem. *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open and $1 \le p < \infty$. Then:*

(i) If $1 \le p < n$, there holds

$$W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$$[14]

with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ and there holds

$$\|u\|_{p^*,\Omega} \le c\|Du\|_{p,\Omega} \quad \forall u \in W_0^{1,p}(\Omega),$$

where $c = c(n,p)$.

(ii) If $\Omega \Subset \mathbb{R}^n$ and $p > n$, there holds

$$W_0^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\bar{\Omega})$$[15]

with $\alpha = 1 - \frac{n}{p}$ and there holds

$$|u|_{0,\alpha,\Omega} \le c\|Du\|_{p,\Omega} \quad \forall u \in W_0^{1,p}(\Omega),$$

where $c = c(n, p, \operatorname{diam}(\Omega))$.

Proof. (i) We show

$$\exists c = c(n,p)\ \forall u \in W_0^{1,p}(\Omega)\colon\ \|u\|_{p^*,\Omega} \le c\|Du\|_{p,\Omega}.$$

It suffices to show this for $u \in C_c^1(\mathbb{R}^n)$. Let first $p = 1$ and $x = (\hat{x}_i, x^i)$ for all i. There hold

$$|u(x)| \le \int_{-\infty}^{x^i} |u_{,i}(\hat{x}_i, t)| dt,$$

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^n \left(\int_{\mathbb{R}} |u_{,i}(\hat{x}_i, t)| dt \right)^{\frac{1}{n-1}}$$

and hence

$$\int_{\mathbb{R}} |u(x)|^{\frac{n}{n-1}} dx^1 \le \left(\int_{\mathbb{R}} |u_{,1}(\hat{x}_1, t)| dt \right)^{\frac{1}{n-1}} \int_{\mathbb{R}} \prod_{i=2}^n \left(\int_{\mathbb{R}} |u_{,i}(\hat{x}_i, t)| dt \right)^{\frac{1}{n-1}} dx^1.$$

The generalized Hölder inequality implies

$$\int_{\mathbb{R}} |u(x)|^{\frac{n}{n-1}} dx^1 \le \left(\int_{\mathbb{R}} |u_{,1}(\hat{x}_1, t)| dt \right)^{\frac{1}{n-1}} \prod_{i=2}^n \left(\int_{\mathbb{R}^2} |u_{,i}(\hat{x}_i, x^i)| dx^i dx^1 \right)^{\frac{1}{n-1}}.$$

For $n = 2$ this already implies

$$\int_{\mathbb{R}^2} |u|^{\frac{n}{n-1}} \le \int_{\mathbb{R}^2} |u_{,1}| \int_{\mathbb{R}^2} |u_{,2}|.$$

---

[14] For linear subspaces V, W of a vector space E, equipped with different norms, the symbol

$$V \hookrightarrow W$$

means that $V \subset W$ and the inclusion map is continuous.

[15] For elements $u \in W_0^{1,p}(\Omega)$ it has to be understood to mean that one function representing u is in $C^{0,\alpha}(\bar{\Omega})$ and satisfies the estimate.

For $n > 2$ we repeat this argument to obtain

$$\int_{\mathbb{R}^2} |u|^{\frac{n}{n-1}} dx^1 dx^2 \leq \left(\int_{\mathbb{R}^2} |u_{,2}(\hat{x}_2, x^2)| dx^2 dx^1 \right)^{\frac{1}{n-1}} \left(\int_{\mathbb{R}^2} |u_{,1}(\hat{x}_1, x^1)| dx^1 dx^2 \right)^{\frac{1}{n-1}} \cdot \prod_{i=3}^{n} \left(\int_{\mathbb{R}^3} |u_{,i}(\hat{x}_i, x^i)| dx^i dx^1 dx^2 \right)^{\frac{1}{n-1}}.$$

Successive integration implies

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |u_{,i}| \right)^{\frac{1}{n-1}} \leq \left(\int_{\mathbb{R}^n} |Du| \right)^{\frac{n}{n-1}}$$

and hence

$$\forall u \in C_c^1(\mathbb{R}^n) \colon \|u\|_{\frac{n}{n-1}, \mathbb{R}^n} \leq \|Du\|_{1, \mathbb{R}^n}.$$

Now let $1 < p < n$: Define

$$t := \frac{p(n-1)}{n-p} > 1.$$

Then

$$v := |u|^t \in C_c^1(\mathbb{R}^n)$$

and applying what we have just proven, we get

$$\int_{\mathbb{R}^n} |v|^{\frac{n}{n-1}} \leq \left(\int_{\mathbb{R}^n} |Dv| \right)^{\frac{n}{n-1}}.$$

We calculate

$$|Dv| \leq t|u|^{\frac{n(p-1)}{n-p}} |Du|$$

and deduce

$$\|v\|_{\frac{n}{n-1}, \mathbb{R}^n} \leq t \int_{\mathbb{R}^n} |u|^{\frac{n(p-1)}{n-p}} |Du| \leq t\|Du\|_p \left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}} \right)^{\frac{p-1}{p}}.$$

Inserting $|u|^t$ gives

$$\|u\|_{p^*, \mathbb{R}^n} \leq t\|Du\|_{p, \mathbb{R}^n}.$$

(ii) We will show

$$\forall u \in W_0^{1,p}(\Omega) \colon |u|_{0,\alpha,\Omega} \leq c\|Du\|_{p,\Omega}.$$

Let $x_1, x_2 \in \Omega$, $0 < \rho = |x_1 - x_2|$ and $x \in B_\rho(\frac{x_1+x_2}{2}) \equiv B_\rho$. Then for $u \in C_c^1(\Omega)$:[16]

$$\begin{aligned} u(x) - u(x_i) &= \int_0^1 \frac{d}{dt} u(x_i + t(x - x_i)) \, dt \\ &\equiv \int_0^1 u_{,k}(x_i + t(x - x_i))(x^k - x_i^k) \, dt \\ &\leq 2\rho \int_0^1 |Du(x_i + t(x - x_i))| \, dt. \end{aligned}$$

[16] extended to $\mathbb{R}^n$ by zero

Thus, with possibly varying constants $c = c(n)$,

$$\begin{aligned}\left|\fint_{B_\rho} u - u(x_i)\right| &\le c\rho^{1-n}\int_0^1\int_{B_\rho}|Du(x_i + t(x - x_i))|\,dxdt\\ &= c\rho^{1-n}\int_0^1 t^{-n}\int_{B_{\rho t}}|Du(z)|\,dzdt\\ &\le c\rho^{1-n}\int_0^1 t^{-n}\rho^{n\frac{p-1}{p}}t^{n\frac{p-1}{p}}\,dt\|Du\|_{p,\mathbb{R}^n}\\ &= c\rho^{1-\frac{n}{p}}\|Du\|_{p,\Omega}\int_0^1 t^{-\frac{n}{p}}\,dt\\ &= c(n,p)\rho^{1-\frac{n}{p}}\|Du\|_{p,\Omega}.\end{aligned}$$

Finally

$$\begin{aligned}|u(x_1) - u(x_2)| &\le \left|u(x_1) - \fint_{B_\rho} u\right| + \left|\fint_{B_\rho} u - u(x_2)\right|\\ &\le c\|Du\|_{p,\Omega}|x_1 - x_2|^\alpha\end{aligned}$$

with $\alpha = 1 - \frac{n}{p}$.

Choosing $x_2 \in \partial\Omega$ we find $u(x_2) = 0$ and thus

$$|u|_{0,\Omega} \le c\|Du\|_{p,\Omega}(\text{diam }\Omega)^\alpha.$$

For $u \in W_0^{1,p}(\Omega)$ choose an approximating sequence $(u_n)_{n\in\mathbb{N}}$ in $C_c^1(\Omega)$, then by the previous estimate $(u_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in $C^{0,\alpha}(\bar{\Omega})$ and has a limit $v \in C^{0,\alpha}(\bar{\Omega})$. Since for almost every x and a subsequence there holds

$$u_n(x) \to u(x),$$

v is a Hölder-continuous representative of u and satisfies the estimate. □

If we relax the target space of these embeddings a little bit, we even obtain compact embeddings.

3.4.2 Lemma (Interpolation). *Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ open.*

(i) If $1 \le p_1 < p < p_2 < \infty$ and

$$\frac{1}{p} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2}, \quad 0 < \alpha < 1,$$

then

$$\|u\|_{p,\Omega} \le \|u\|_{p_1,\Omega}^\alpha\|u\|_{p_2,\Omega}^{1-\alpha} \quad \forall u \in L^{p_1}(\Omega) \cap L^{p_2}(\Omega).$$

(ii) If $0 < \beta < \alpha \le 1$, then

$$[u]_{\beta,\Omega} \le 2^{1-\frac{\beta}{\alpha}}[u]_{\alpha,\Omega}^{\frac{\beta}{\alpha}}|u|_{0,\Omega}^{1-\frac{\beta}{\alpha}} \quad \forall u \in C^{0,\alpha}(\bar{\Omega}).$$

Proof. (i) There holds

$$p = \frac{1}{\alpha p_2 + (1-\alpha)p_1}(\alpha p_1 p_2 + (1-\alpha)p_1 p_2).$$

Thus

$$\begin{aligned}\int_\Omega |u|^p &= \int_\Omega |u|^{p_1 \frac{\alpha p_2}{\alpha p_2+(1-\alpha)p_1}} |u|^{p_2 \frac{(1-\alpha)p_1}{\alpha p_2+(1-\alpha)p_1}} \\ &\leq \left(\int_\Omega |u|^{p_1}\right)^{\frac{\alpha p_2}{\alpha p_2+(1-\alpha)p_1}} \left(\int_\Omega |u|^{p_2}\right)^{\frac{(1-\alpha)p_1}{\alpha p_2+(1-\alpha)p_1}}.\end{aligned}$$

(ii)

$$\begin{aligned}\frac{|u(x)-u(y)|}{|x-y|^\beta} &= \left(\frac{|u(x)-u(y)|^{\frac{\alpha}{\beta}}}{|x-y|^\alpha}\right)^{\frac{\beta}{\alpha}} \\ &= \left(\frac{|u(x)-u(y)|}{|x-y|^\alpha}|u(x)-u(y)|^{\frac{\alpha}{\beta}-1}\right)^{\frac{\beta}{\alpha}} \\ &\leq 2^{1-\frac{\beta}{\alpha}}[u]_{\alpha,\Omega}^{\frac{\beta}{\alpha}}|u|_{0,\Omega}^{1-\frac{\beta}{\alpha}}.\end{aligned}$$

□

3.4.3 Theorem. *Let $n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ open and $1 \leq p < \infty$.*

(i) If $1 \leq p < n$, then the embedding

$$W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

with $\frac{1}{q} > \frac{1}{p} - \frac{1}{n}$ is compact.

(ii) If $p > n$, then

$$W_0^{1,p}(\Omega) \hookrightarrow C^{0,\beta}(\bar{\Omega})$$

with $\beta < 1 - \frac{n}{p}$ is compact.

Proof. Due to Corollary 3.1.18, Theorem 3.4.1 and Lemma 3.4.2 the second claim is true. To prove (i), suppose that $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W_0^{1,p}(\Omega)$. Choose a sequence $(v_k)_{k\in\mathbb{N}}$ in $C_c^\infty(\Omega)$, such that

$$\|u_k - v_k\|_{1,p,\Omega} < \frac{1}{k}.$$

It suffices to prove that $(v_k)_{k\in\mathbb{N}}$ has a convergent subsequence in $L^q(\Omega)$ and by Lemma 3.4.2 it suffices to show this for $q = 1$. We use the Kolmogorov characterisation, Theorem 3.1.19. The boundedness already holds in $L^{p^*}(\Omega)$ and thus also in $L^1(\Omega)$. We prove the continuity in the mean.

$$\begin{aligned}v_k(x+h) - v_k(x) &= \int_0^1 \frac{d}{dt} v_k(x+th)dt \\ &= \int_0^1 v_{k,i}(x+th)h^i dt\end{aligned}$$

and thus

$$\limsup_{h\to 0} \int_{\mathbb{R}^n} |v_k(x+h) - v_k(x)|\, dx \leq \limsup_{h\to 0} |h| \int_0^1 \int_{\mathbb{R}^n} |Dv_k| = 0.$$

□

Compactness theorems of higher order follow and shall be proved as an exercise.

3.4.4 Exercise. Let $n, m \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ open and $1 \leq p < \infty$. Then there hold

(i) If $mp < n$ and $q < \frac{np}{n-mp}$, then

$$W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$$

is compact.

(ii) If $0 \leq k < m - \frac{n}{p}$, then

$$W_0^{m,p}(\Omega) \hookrightarrow C^k(\bar{\Omega})$$

is compact.

Due to its importance, also historically, let us write down a corollary of the previous compactness theorem, which is known as *Rellich's embedding theorem.*

3.4.5 Theorem (Rellich). *Let $n, m \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ open and $1 \leq p < \infty$. Then*

$$W_0^{m,p}(\Omega) \hookrightarrow W_0^{m-1,p}(\Omega)$$

is compact.

Proof. Induction, $m = 1$. If $p \leq n$, then

$$W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,p-\epsilon}(\Omega) \hookrightarrow L^{\frac{n(p-\epsilon)}{n-(p-\epsilon)}-\epsilon}(\Omega)$$

is compact for small ϵ and there holds

$$p < \frac{n(p-\epsilon)}{n-(p-\epsilon)} - \epsilon.$$

If $p > n$, then

$$W_0^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{n}{p}-\epsilon}(\bar{\Omega}) \hookrightarrow L^p(\Omega)$$

is compact for small ϵ. Suppose the result is true for $m \geq 1$, then

$$D(W_0^{m+1,p}(\Omega)) \hookrightarrow W_0^{m,p}(\Omega) \hookrightarrow W_0^{m-1,p}(\Omega)$$

is compact. So let $(u_k)_{k\in\mathbb{N}}$ be bounded in $W_0^{m+1,p}(\Omega)$. Then $(u_k)_{k\in\mathbb{N}}$ and $(Du_k)_{k\in\mathbb{N}}$ are bounded in $W_0^{m,p}(\Omega)$ and hence a subsequence is a Cauchy sequence in $W^{m-1,p}(\Omega)$

$$\|u_k - u_l\|_{m-1,p,\Omega} \to 0, \quad \|Du_k - Du_l\|_{m-1,p,\Omega} \to 0.$$

Then

$$\|u_k - u_l\|_{m,p,\Omega}^p = \sum_{\langle\alpha\rangle \leq m} \|(u_k - u_l)_{,\alpha}\|_{p,\Omega}^p \to 0.$$

□

3.5 Extension of Sobolev functions

Since in the end we want to solve the Dirichlet problem, we have to discuss boundary values of Sobolev functions. We prepare this with several results, amongst which there is a generalisation of the embedding theorems.

3.5.1 Lemma. *Let $n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ open with C^m-boundary or $\Omega = \mathbb{R}^n_+$ and $1 \leq p < \infty$. Then $C^\infty(\bar{\Omega})$ is dense in $W^{m,p}(\Omega)$.*

Proof. Cover $\partial\Omega$ by finitely many open sets $(U_i)_{1\leq i \leq N}$, which lie in the domains of local straightening functions $(\psi_i)_{1\leq i\leq N}$ with image $\mathbb{R}^n_+$ and define

$$U_0 = \Omega \backslash \bigcup_{k=1}^{N} \bar{U}_k.$$

Choose a finite partition of unity $(\eta_j)_{1\leq j\leq m}$ for

$$(U_i)_{0\leq i\leq N}\,,$$

where

$$\operatorname{supp}\eta_j \subset U_0, \quad 1 \leq j \leq l$$

and

$$\operatorname{supp}\eta_j \subset U_i, \quad l+1 \leq j \leq m$$

for suitable i. It suffices to prove that

$$w := w_j = u\eta_j \circ \psi_i^{-1}$$

can be approximated by functions $(f^j_k)_{k\in\mathbb{N}}$ in $C^\infty(\bar{\mathbb{R}}^n_+)$, since in this case we first pick a sequence $(g_k)_{k\in\mathbb{N}}$ in $C^\infty_c(\Omega)$ with

$$g_k \to \sum_{j=1}^{l} \eta_j u \in W^{m,p}_0(\Omega)$$

and then we calculate with the help of Proposition 3.3.7:

$$\begin{aligned}
&\|u - g_k + \sum_{j=l+1}^{m} \eta_j f^j_k \circ \psi_i\|_{m,p,\Omega} \\
\leq\ & \|\sum_{j=1}^{l} \eta_j u - g_k\|_{m,p,\Omega} + \|\sum_{j=l+1}^{m} (\eta_j u - \eta_j f^j_k \circ \psi^i)\|_{m,p,\Omega} \\
\to\ & 0
\end{aligned}$$

for $k \to \infty$. So let us prove that $w \in W^{m,p}(\mathbb{R}^n_+)$ can be approximated by a sequence $(f_k)_{k\in\mathbb{N}}$ in $C^\infty(\bar{\mathbb{R}}^n_+)$. Define for $h > 0$,

$$w_h(x) = w(x + 2he_n),$$

then $w_h \in W^{m,p}(\{x^n > -2h\})$. For small ϵ the convolutions

$$w^\epsilon_h \in C^\infty\left(\overline{\{x^n > -h\}}\right)$$

approximate w_h in $W^{m,p}(\{x^n > -h\})$. Furthermore $w_h \to w$ in $W^{m,p}(\mathbb{R}^n_+)$, since L^p-functions are equicontinuous in the mean. □

3.5.2 Lemma (Lions-Magenes). *Let $c_1, ..., c_{m+1}$ be solutions to the linear system*

$$\sum_{k=1}^{m+1} (-1)^j k^j c_k = 1, \quad 0 \le j \le m.$$

For $u \in W^{m,p}(\mathbb{R}^n_+) \cap C^\infty(\bar{\mathbb{R}}^n_+)$,

$$\tilde{u}(\hat{x}, x^n) = \sum_{k=1}^{m+1} c_k u(\hat{x}, -kx^n), \quad x^n < 0,$$

defines an extension of u to $\mathbb{R}^n$, such that $\tilde{u} \in C^m(\mathbb{R}^n)$ and

$$\|\tilde{u}\|_{m,p,\mathbb{R}^n} \le c\|u\|_{m,p,\mathbb{R}^n_+}, \quad c = c(m, n, p), \quad 1 \le p \le \infty.$$

Proof. Let $x \in \mathbb{R}^n_-$. Then there holds

$$D\tilde{u}(\hat{x}, x^n) = \left(\sum_{k=1}^{m+1} c_k \frac{\partial u}{\partial x^1} u(\hat{x}, -kx^n), ..., \sum_{k=1}^{m+1} (-k) c_k \frac{\partial u}{\partial x^n}(\hat{x}, -kx^n) \right).$$

Hence

$$\tilde{u} \in C^1(\bar{\mathbb{R}}^n_-)$$

and

$$\lim_{x^n \to 0+} D\tilde{u}(\hat{x}, x^n) = \lim_{x^n \to 0-} D\tilde{u}(\hat{x}, x^n).$$

This implies $\tilde{u} \in C^1(\mathbb{R}^n)$. In exactly the same way one can iterate this process up to m derivatives to show that $\tilde{u} \in C^m(\mathbb{R}^n)$. To prove the estimate, calculate for an arbitrary multi-index $|\beta| \le m$:

$$\begin{aligned}
\|\tilde{u}_{,\beta}\|^p_{m,p,\mathbb{R}^n} &= \|u_\beta\|^p_{m,p,\mathbb{R}^n_+} + \int_{\mathbb{R}^n_-} \left| \left(\sum_{k=1}^{m+1} c_k u(\hat{x}, -kx^n) \right)_{,\beta} \right|^p dx \\
&\le \|u_{,\beta}\|^p_{m,p,\mathbb{R}^n_+} + \sum_{k=1}^{m+1} c_k k^{mp} \int_{\mathbb{R}^n_-} |u_{,\beta}(\hat{x}, -kx^n)|^p dx \\
&\le c(m, n, p)\|u\|_{m,p,\mathbb{R}^n_+}.
\end{aligned}$$

□

Using a partition of unity, we can prove an extension theorem for Sobolev functions.

3.5.3 Theorem (Extension of Sobolev functions). *Let $n, m \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ open with C^m-boundary and $1 \le p < \infty$. Then for any open set Ω_0 with $\Omega \Subset \Omega_0$ there exists a bounded linear extension operator*

$$E \colon W^{m,p}(\Omega) \to W^{m,p}_0(\Omega_0),$$

such that

$$Eu_{|\Omega} = u$$

and

$$\|Eu\|_{m,p,\Omega_0} \le c\|u\|_{m,p,\Omega},$$

where $c = c(n, m, p, \partial\Omega, \mathrm{dist}(\Omega, \partial\Omega_0))$.

Proof. We may assume $u \in C^\infty(\bar{\Omega})$. Cover $\partial\Omega$ by finitely many open sets $(U_i)_{1\le i\le N}$ with $U_i \Subset \Omega_0$, which lie in the domains of local straightening functions $(\psi_i)_{1\le i\le N}$ with image $\mathbb{R}^n_+$ and choose a finite partition of unity $(\eta_j)_{1\le j\le m}$ for

$$\left(U_i, \Omega\backslash \bigcup_{k=1}^{N} \bar{U}_k\right)_{1\le i\le N}.$$

Then for each j, $\operatorname{supp}(u\eta_j) \subset U_i$ for some i. Hence, defining

$$E(u\eta_j \circ \psi_i^{-1}) \in W^{m,p}(\mathbb{R}^n)^{17}$$

to be the Lions-Magenes extension of $u\eta_j \circ \psi_i^{-1}$, we deduce

$$\|E(u\eta_j \circ \psi_i^{-1})\|_{m,p,\mathbb{R}^n} \le c\|u\eta_j \circ \psi_i^{-1}\|_{m,p,\mathbb{R}^n_+}.$$

Let $1 \le j \le k$ be those indices with

$$\operatorname{supp} \eta_j \subset \Omega\backslash \bigcup_{r=1}^{N} \bar{U}_r.$$

Define

$$Eu = \sum_{j=1}^{k} \eta_j u + \sum_{j=k+1}^{m} E(u\eta_j \circ \psi_i^{-1}) \circ \psi_i.$$

Then $Eu \in W_0^{m,p}(\Omega_0)$, $Eu_{|\Omega} = u$ and

$$\|Eu\|_{m,p,\Omega_0} \le c\|u\|_{m,p,\Omega}.$$

□

3.5.4 Corollary. *Let $n, m \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ open with C^m-boundary and $1 \le p < \infty$. Then there hold*

(i) If $mp < n$ and $q < \frac{np}{n-mp}$, then

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$$

is compact.

(ii) If $0 \le k < m - \frac{n}{p}$, then

$$W^{m,p}(\Omega) \hookrightarrow C^k(\bar{\Omega})$$

is compact.

Proof.

$$W^{m,p}(\Omega) \xrightarrow{E} W_0^{m,p}(\Omega_0) \hookrightarrow L^q(\Omega_0) \xrightarrow{\cdot_{|\Omega}} L^q(\Omega)$$

is compact, and similarly for the embedding into C^k. □

[17] Note that E can be continuously extended to $W^{m,p}(\mathbb{R}^n_+)$.

Chapter 4

Elliptic existence and regularity theory for weak solutions

4.1 Weak solutions to linear equations

Following our previous philosophy that we should search for a solution of, e.g.

$$\Delta u = f \tag{4.1}$$

in a larger function space, this equation must of course be understood in a weak sense. The broadest sense that we have considered so far is distributional i.e. we should consider (4.1) to be defined by

$$\forall \varphi \in C_c^\infty(\Omega) \colon \int_\Omega u \Delta \varphi = \int_\Omega f \varphi,$$

where $f \in L^1_{\mathrm{loc}}(\Omega)$. In principle we could try to find a solution $u \in L^1_{\mathrm{loc}}(\Omega)$ to this equation. However, we already announced, that we want to use the elegant Hilbert space method. Since $L^1_{\mathrm{loc}}(\Omega)$ is not a Hilbert space, we must modify the setting a bit and hence we will allow $u \in W_0^{1,2}(\Omega)$.

This modification also includes a new kind of differential operator. In the linear theory we have so far considered elliptic operators of the form

$$Lu = a^{ij} u_{,ij} + b^i u_{,i} + du.$$

Since now we only allow $u \in W_0^{1,2}(\Omega)$, the only way to make sense of this is the distributional one, i.e. it has to be understood as

$$\int_\Omega \varphi L u = -\int_\Omega (a^{ij}\varphi)_{,j} u_{,i} + \int_\Omega b^i u_{,i} \varphi + \int_\Omega du\varphi.$$

Since this form would require some regularity of a^{ij}, which we do not want to assume in general, it is more convenient to work with the following structure.

4.1.1 Definition (Divergence form operator). Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ open.

(i) Let
$$\Gamma \subset \mathbb{R}^n \times \mathbb{R} \times \Omega.$$
A *divergence form partial differential operator of second order in* Ω is a map
$$\mathcal{L}_{A,B} \colon \mathcal{A} \subset W^{1,2}(\Omega) \to \mathcal{D}'(\Omega)$$
$$u \mapsto (A^i(Du,u,\cdot))_{,i} + B(Du,u,\cdot),$$
where for $1 \leq i \leq n$,
$$A^i, B \colon \Gamma \to \mathbb{R}$$
and
$$\mathcal{A} = \{u \in W^{1,2}(\Omega) \colon (Du(x),u(x),x) \in \Gamma \text{ for a.e. } x \in \Omega, \\ A^i(Du,u,\cdot), B(Du,u,\cdot) \in L^1_{\text{loc}}(\Omega)\}$$ [1]
is the set of (A,B)-*admissable* functions.

(ii) $\mathcal{L}_{A,B}$ is called *elliptic in* $u \in \mathcal{A}$, if
$$\left(A^{ij}(Du(x),u(x),x)\right) := \left(\frac{\partial A^i}{\partial p_j}(Du(x),u(x),x)\right)_{\text{sym}}$$
exists and is positive definite for almost every $x \in \Omega$. For a set $\mathcal{S} \subset \mathcal{A}$, $\mathcal{L}_{A,B}$ is called *elliptic operator in* $\mathcal{S}$, if $\mathcal{L}_{A,B}$ is elliptic in all $u \in \mathcal{S}$.

(iii) Let $\mathcal{S} \subset \mathcal{A}$. $\mathcal{L}_{A,B}$ is called *strictly elliptic in* $\mathcal{S}$, if
$$\exists \lambda > 0 \; \forall u \in \mathcal{S} \; \forall (\xi_i) \in \mathbb{R}^n \colon A^{ij}(Du,u,\cdot)\xi_i\xi_j \geq \lambda|\xi|^2$$
and *uniformly elliptic in* S, if
$$\exists 0 < \lambda < \Lambda \; \forall u \in \mathcal{S} \; \forall (\xi_i) \in \mathbb{R}^n \colon \lambda|\xi|^2 \leq A^{ij}(Du,u,\cdot)\xi_i\xi_j \leq \Lambda|\xi|^2.$$

4.1.2 Example. (i) With
$$A(Du) = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}},$$
we see that the minimal surface operator is in divergence form.

(ii) A general linear divergence form operator has the form
$$Lu = (a^{ij}u_{,j} + a^i u)_{,i} + b^i u_{,i} + du$$
with coefficients $a^{ij}, a^i, b^i, d \in L^2_{\text{loc}}(\Omega)$, $1 \leq i,j \leq n$. However, we will later assume coefficients in $L^\infty(\Omega)$ in order to ensure that L maps into the dual space of $W^{1,2}_0(\Omega)$.

We can now prove an existence and uniqueness result. For the uniqueness we have already seen that we have to impose boundary conditions in general. This is incorporated in the weak setting by restricting the domain to $W^{1,2}_0(\Omega)$. We follow [5, Ch. 8].

[1] Recall how an $L^1_{\text{loc}}(\Omega)$-function acts as a distribution.

4.1.3 Lemma (Maximum principle for weak solutions). *Let $n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ open and*

$$L\colon W_0^{1,2}(\Omega) \to \mathcal{D}'(\Omega)$$
$$Lu = (a^{ij}u_{,j} + a^i u)_{,i} + b^i u_{,i} + du$$

with coefficients in $L^\infty(\Omega)$,

$$\int_\Omega (d\varphi - a^i\varphi_{,i}) \leq 0 \quad \forall 0 \leq \varphi \in C_c^\infty(\Omega)$$

and

$$\forall (\xi_i) \in \mathbb{R}^n \colon a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$$

with $\lambda > 0$. Then L is injective.

Proof. Step 1: We show that for $u \in W_0^{1,2}(\Omega)$ and any $k > 0$ there holds

$$v_k := \max(u - k, 0) \in W_0^{1,2}(\Omega).$$

Therefore choose a $(\varphi_m)_{m\in\mathbb{N}}$ of functions in $C_c^\infty(\Omega)$ which approximate u in $W^{1,2}(\Omega)$ and pointwise almost everywhere. Due to Lemma 3.3.8 the functions

$$\psi_m = \max(\varphi_m - k, 0)$$

are in $W^{1,2}(\Omega)$ and have compact support in Ω. Furthermore they approximate v_k:

$$\|\psi_m - v_k\|_{2,\Omega}^2 \leq \int_\Omega |u - \varphi_m|^2 \to 0$$

and

$$\begin{aligned}
&\int_\Omega |D\left(\max(u-k,0) - \max(\varphi_m - k, 0)\right)|^2 \\
= &\int_\Omega |Du\chi_{\{u>k\}} - D\varphi_m\chi_{\{\varphi_m>k\}}|^2 \\
= &\int_\Omega |(Du - D\varphi_m)\chi_{\{\varphi_m>k\}} + Du(\chi_{\{u>k\}} - \chi_{\{\varphi_m>k\}})|^2 \\
\to\ & 0
\end{aligned}$$

due to the dominated convergence theorem.

Step 2: We prove that L is injective, so let $Lu = 0$. Then for all $k > 0$

$$\begin{aligned}
0 = Lu(v_k) &= -\int_\Omega (a^{ij}u_{,i}v_{k,j} + a^i v_{k,i}u) + \int_\Omega (b^i u_{,i}v_k + duv_k) \\
&= -\int_\Omega a^{ij}v_{k,i}v_{k,j} + \int_\Omega (a^i + b^i)u_{,i}v_k + \int_\Omega (duv_k - a^i(uv_k)_{,i}) \quad (4.2)\\
&\leq -\lambda\int_\Omega |Dv_k|^2 + c\|v_k\|_{2,\{u>k\}}\|Dv_k\|_{2,\Omega}
\end{aligned}$$

and hence

$$\|Dv_k\|_{2,\Omega} \leq c\|v_k\|_{2,\{u>k\}}. \tag{4.3}$$

Applying the Sobolev embedding theorem in case $n \geq 3$, we obtain

$$\|v_k\|_{\frac{2n}{n-2},\Omega} \leq c\|v_k\|_{2,\{u>k\}} \leq c\|v_k\|_{\frac{2n}{n-2},\Omega}\mathcal{L}^n(\{u > k\})^{\frac{1}{n}}.$$

Assuming $v_k \neq 0$ for some k, we obtain

$$\mathcal{L}^n(\{u > k\}) \geq c^{-n}.$$

This implies that u must be bounded, since if it was unbounded, then $v_k \neq 0$ for all k and

$$\int_\Omega |u|^2 \geq \int_{\{u>k\}} |u|^2 \geq c^{-n}k^2.$$

But this implies $u \notin L^2(\Omega)$, a contradiction. Starting from (4.2) we can repeat this calculation with $\{u > k\}$ replaced by $\{Dv_k \neq 0\}$ and obtain for all $0 < k < \sup u$ that

$$\mathcal{L}^n(\{Dv_k \neq 0\}) \geq c^{-n}.$$

Since $\{Dv_k \neq 0\} \subset \{u > k\}$ and $Dv_k = 0$ almost everywhere on $\{u = \sup u\}$, we obtain

$$c^{-n} \leq \mathcal{L}^n(\{Dv_k \neq 0\}) \leq \mathcal{L}^n(\{k < u < \sup u\}) \to 0,$$

as $k \to \sup u$, contradiction. Hence $v_k = 0$ for all k and thus $u \leq 0$.

Step 3: $n = 2$. Starting from (4.3) and applying the Sobolev embedding with some $2 - \epsilon < p < 2$ for small ϵ, we get

$$\|v_k\|_{p^*,\Omega} \leq c\|Dv_k\|_{p,\Omega} \leq c\|Dv_k\|_{2,\Omega} \leq c\|v_k\|_{2,\{u>k\}} \leq c\|v_k\|_{p^*,\Omega}\mathcal{L}^n(\{u > k\})^c$$

and the proof can be continued as in case $n \geq 3$.

Step 4: $n = 1$. Again from (4.3) we obtain, using the embedding into Hölder spaces,

$$|v_k|_{0,\alpha,\Omega} \leq c\|Dv_k\|_{2,\Omega} \leq c\left(\int_{\{u>k\}} |v_k|^2\right)^{\frac{1}{2}} \leq c|v_k|_{0,\alpha,\Omega}\mathcal{L}^n(\{u > k\})^{\frac{1}{2}}$$

and the proof can be finished as in the previous steps.

This proves $u \leq 0$ in any of the cases and by replacing u by $-u$, we obtain $u = 0$. □

4.1.4 Theorem (Existence of weak solutions). *Let $n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ open and*

$$L\colon W_0^{1,2}(\Omega) \to \mathcal{D}'(\Omega)$$
$$Lu = (a^{ij}u_{,j} + a^i u)_{,i} + b^i u_{,i} + du$$

with coefficients in $L^\infty(\Omega)$,

$$\int_\Omega (d\varphi - a^i\varphi_{,i}) \leq 0 \quad \forall 0 \leq \varphi \in C_c^\infty(\Omega)$$

and

$$\forall (\xi_i) \in \mathbb{R}^n\colon a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$$

with $\lambda > 0$. Then L is a continuous linear isomorphism onto $W_0^{1,2}(\Omega)'$ with continuous inverse.

Proof. Step 1: First we prove this for a modified operator

$$\tilde{L}u = Lu - \sigma u,$$

where a sufficiently large $\sigma \in \mathbb{R}$ will be chosen later. The map

$$\tilde{L}u \in \mathcal{D}'(\Omega)$$

is given by

$$\tilde{L}u(\varphi) = -\int_\Omega (a^{ij}u_{,j} + a^i u)\varphi_{,i} + \int_\Omega (b^i u_{,i} + du)\varphi - \int_\Omega \sigma u\varphi$$

and hence

$$|\tilde{L}u(\varphi)| \leq (c + \sigma)\|u\|_{1,2,\Omega}\|\varphi\|_{1,2,\Omega}. \tag{4.4}$$

Hence the map Lu, restricted to the dense subspace $(C_c^\infty(\Omega), \|\cdot\|_{1,2,\Omega})$ is continuous and thus extends uniquely to an element $\tilde{L}u \in W_0^{1,2}(\Omega)'$. Now define a bilinear form on $W_0^{1,2}(\Omega)$ by

$$B(v, u) = \tilde{L}u(v).$$

Since (4.4) carries over to $v \in W_0^{1,2}(\Omega)$, B is bounded as a bilinear form. Furthermore $-B$ is coercive, provided σ is large enough:

$$\begin{aligned}
-B(u, u) &= \int_\Omega a^{ij}u_{,i}u_{,j} + (a^i - b^i)u_{,i}u - \int_\Omega (d - \sigma)u^2 \\
&\geq \lambda\|Du\|_{2,\Omega} - \|a - b\|_{\infty,\Omega}\left(\frac{\epsilon}{2}\|Du\|_{2,\Omega} + \frac{1}{2\epsilon}\|u\|_{2,\Omega}\right) \\
&\quad - (\|d\|_{\infty,\Omega} - \sigma)\|u\|_{2,\Omega} \\
&\geq \frac{\lambda}{2}\|Du\|_{2,\Omega} + \frac{\lambda}{2}\|u\|_{2,\Omega} \\
&\geq \frac{c\lambda}{2}\|u\|_{1,2,\Omega},
\end{aligned}$$

provided that first ϵ is chosen small enough (in dependence of the data of the problem) and then σ is chosen large enough. Due to Theorem 3.1.16, for every $\psi \in W_0^{1,2}(\Omega)'$ there exists a unique $u \in W_0^{1,2}(\Omega)$, such that for all $\varphi \in W_0^{1,2}(\Omega)$

$$\tilde{L}(u)\varphi = B(\varphi, u) = \psi(\varphi).$$

Hence $\tilde{L}$ maps bijectively to $W_0^{1,2}(\Omega)$ and due to the coercivity and the boundedness of $-B$ this map is also continuous with continuous inverse.

Step 2: We show that the map we added,

$$\begin{aligned}
I\colon W_0^{1,2}(\Omega) &\to W_0^{1,2}(\Omega)' \\
u &\mapsto I(u) = \int_\Omega u\cdot,
\end{aligned}$$

is compact. But I is just the restriction of the corresponding map defined on $L^2(\Omega)$ to the compactly embedded[2] subspace $W_0^{1,2}(\Omega)$. Hence I is compact.

[2] Rellich, Theorem 3.4.5

Step 3: Now we prove the claim of the theorem for L. Unique solvability of $Lu = \psi$ is equivalent to unique solvability of

$$\tilde{L}u + \sigma I u = \psi,$$

which is in turn equivalent to unique solvability of

$$u + \sigma \tilde{L}^{-1} I u = \tilde{L}^{-1}\psi.$$

Since $\tilde{L}^{-1} \circ I$ is compact, the Fredholm-alternative, Theorem 3.1.14, says that $\mathrm{id} + (\sigma \tilde{L}^{-1} \circ I)$ is surjective if and only if it is injective. However, the uniqueness of solutions of

$$u + \sigma \tilde{L}^{-1} \circ I = 0$$

follows from the uniqueness of solutions of $Lu = 0$, Lemma 4.1.3. □

4.1.5 Remark (Sturm-Liouville problem). Since we did not restrict the dimension, the previous results contain a partial solution to the so-called *Sturm-Liouville problem*, which is a Dirichlet problem for second order ordinary differential equations: On an interval $I = [a, b]$ let three functions $p, q \in L^\infty(a, b)$ and $w \in C^1([a, b])$ be given, such that

$$p \geq c > 0, \quad w > 0.$$

The Sturm-Liouville problem asks to find pairs (u, λ), which solve the eigenvalue problem

$$\begin{aligned} -(pu')' + qu &= \lambda w u \\ u(a) = u(b) &= 0. \end{aligned}$$

Since $w \in C^1([a, b])$, we can rewrite this equation to

$$-\left(\frac{p}{w}u'\right)' - \frac{p}{w^2}w'u' + \frac{q}{w}u = \lambda u$$

and we see from Theorem 4.1.4, that for

$$\lambda \leq \min_{x \in I} \frac{q(x)}{w(x)}$$

there are no nonzero solutions. Later we will also prove the existence of nonvanishing solution for certain λ which violate this condition.

4.2 Regularity of weak solutions

We achieved existence and uniqueness of a solution $u \in W_0^{1,2}(\Omega)$ to

$$Lu = f$$

for a large class of right hand sides, namely for all $f \in W_0^{1,2}(\Omega)'$. The aim of this section is to deduce higher regularity of u, once that f is more regular. These estimates divide into *interior estimates* and *boundary estimates*. For the case $f \in L^2(\Omega)$ we will not only achieve this for the linear operator L we have treated in the previous section, but for more general nonlinear divergence form operators, since the proof does not essentially make use of a linear structure.

Interior estimates

4.2.1 Theorem (Interior $W^{2,2}$-estimate)**.** *Let $n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ be open and $f \in L^2_{\mathrm{loc}}(\Omega)$. Let $A \in C^1(\mathbb{R}^n \times \mathbb{R} \times \Omega, \mathbb{R}^n)$ satisfy*

$$|\partial_x A(p,z,x)| \le c_A^1(c_A^2 + |p| + |z|), \quad |\partial_z A| + |\partial_p A| \le c_A^1$$

for some $c_A^i > 0$. Suppose $B \in \mathbb{R}^{\mathbb{R}^n \times \mathbb{R} \times \Omega}$ is measurable and satisfies for almost every (p,z,x),

$$|B(p,z,x)| \le c_B^1(c_B^2 + |p| + |z|).$$

Let

$$\mathcal{L}_{A,B} \colon W_0^{1,2}(\Omega) \to \mathcal{D}'(\Omega)$$

be a divergence form partial differential operator of second order in Ω, strictly elliptic in $W_0^{1,2}(\Omega)$ with ellipticity constant $\lambda_0 > 0$. Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a solution of the distributional equality

$$-\operatorname{div} A(Du,u,\cdot) + B(Du,u,\cdot) = f.$$

Then $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ and for all $\Omega' \Subset \Omega'' \Subset \Omega$ there exists c depending on n, c_A^1, c_B^1, λ_0, $\mathcal{L}^n(\Omega)$ and $\operatorname{dist}(\Omega', \partial\Omega'')$, such that

$$\|u\|_{2,2,\Omega'} \le c(c_A^2 + c_B^2 + \|u\|_{1,2,\Omega''} + \|f\|_{2,\Omega''}).$$

Proof. Let $\Omega' \Subset \Omega'' \Subset \Omega$ and $\Delta_h^i u$ be the difference quotient of stepsize h in direction $1 \le k \le n$,

$$\Delta_h^k u(x) = \frac{u(x + he_k) - u(x)}{h}, \quad |h| < \operatorname{dist}(\Omega', \partial\Omega'').$$

Due to the structure conditions of A there holds

$$A(Du,u,\cdot), B(Du,u,\cdot) \in L^2(\Omega). \tag{4.5}$$

We rewrite the difference quotient of the functions $A^i(Du,u,\cdot)$,

$$\begin{aligned} &\Delta_h^k A^i(Du,u,\cdot) \\ =\ &\frac{1}{h}\int_0^1 \frac{d}{dt} A^i(tDu(\cdot + he_k) + (1-t)Du, tu(\cdot + he_k) + (1-t)u, \cdot + the_k)\, dt \\ =\ &A^{ij}(\Delta_h^k u)_{,j} + a^i \Delta_h^k u + \alpha_k^i, \end{aligned} \tag{4.6}$$

where

$$A^{ij} = \int_0^1 \frac{\partial A^i}{\partial p_j}(tDu(\cdot + h) + (1-t)Du, tu(\cdot + h) + (1-t)u, \cdot + th)\, dt \tag{4.7}$$

and

$$a^i = \int_0^1 \frac{\partial A^i}{\partial z}, \quad \alpha_k^i = \int_0^1 \frac{\partial A^i}{\partial x^k},$$

where the integrand terms are also evaluated at the convex combinations as in (4.7).

Let $\eta \in C_c^\infty(\Omega'')$, such that

$$\eta_{|\Omega'} = 1.$$ [3]

For $|h| < \min(\operatorname{dist}(\Omega'', \partial\Omega), \operatorname{dist}(\operatorname{supp}\eta, \partial\Omega''))$ choose the test function

$$v = -\Delta_{-h}^k(\eta^2 \Delta_h^k u) \in W_0^{1,2}(\Omega)$$

in the equality

$$\int_\Omega A^i(Du, u, \cdot)v_{,i} + \int_\Omega B(Du, u, \cdot)v = \int_\Omega fv.$$ [4]

There holds

$$\int_\Omega \eta^2 \Delta_h^k A^i (\Delta_h^k u)_{,i} = -\int_\Omega 2\eta\eta_{,i}\Delta_h^k A^i \Delta_h^k u - \int_\Omega (f - B)\Delta_{-h}^k(\eta^2\Delta_h^k u)$$

and hence by (4.6) we have for small $\epsilon > 0$ that

$$\begin{aligned}
&\int_\Omega \eta^2 A^{ij}(\Delta_h^k u)_{,i}(\Delta_h^k u)_{,j} \\
= \; & -\int_\Omega 2\eta\eta_{,i}A^{ij}(\Delta_h^k u)_{,j}\Delta_h^k u - \int_\Omega a^i \Delta_h^k u(\eta^2(\Delta_h^k u)_{,i} + 2\eta\eta_{,i}\Delta_h^k u) \\
& -\int_\Omega \alpha_k^i(\eta^2(\Delta_h^k u)_{,i} + 2\eta\eta_{,i}\Delta_h^k u) - \int_\Omega (f-B)\Delta_{-h}^k(\eta^2\Delta_h^k u) \\
\leq \; & \int_\Omega \frac{\eta^2}{2} A^{ij}(\Delta_h^k u)_{,i}(\Delta_h^k u)_{,j} + 2\int_\Omega A^{ij}\eta_{,i}\eta_{,j}(\Delta_h^k u)^2 \\
& + \frac{c\epsilon}{2}\int_\Omega \eta^2 |D(\Delta_h^k u)|^2 + \frac{c}{2\epsilon}\int_{\operatorname{supp}\eta} (c_A^2 + |u|^2 + |Du|^2 + (\Delta_h^k u)^2) \\
& + \frac{\epsilon}{2}\int_\Omega |D(\eta^2\Delta_h^k u)|^2 + \frac{1}{2\epsilon}\int_{\Omega''} |f|^2 + \frac{1}{2\epsilon}\int_{\Omega''} |B(Du, u, \cdot)|^2,
\end{aligned} \tag{4.8}$$

where $c = c(c_A, |\Omega|, \lambda_0, \operatorname{dist}(\Omega', \partial\Omega))$. Due to the strict ellipticity we may absorb and term on the right hand side which contains $D(\Delta_h^k u)$, if we choose $\epsilon > 0$ small enough. Thus (4.8) implies

$$\|D(\Delta_h^k u)\|_{2,\Omega'} \leq c(c_A^2 + c_B^2 + \|u\|_{1,2,\Omega''} + \|f\|_{2,\Omega''}),$$

where $c = c(c_A, \lambda_0, c_B, \mathcal{L}^n(\Omega), \operatorname{dist}(\Omega', \partial\Omega''))$. This completes the proof in view of Lemma 3.3.14. □

4.2.2 Remark. For this illustration we suppose that $B = 0$. The next step would be to deduce higher order estimates, once we know that the data of the problem, A, f are more regular. The general strategy to accomplish this is to apply the equation

$$-(A^i(Du, u, \cdot))_{,i} = f$$

[3] Existence follows from Theorem 1.3.7.

[4] Due to (4.5) the distributional equality carries over to functions in $W_0^{1,2}(\Omega)$.

to a test function of the form

$$\eta = \varphi_{,j},$$

where $\varphi \in C^\infty_c(\Omega)$. Since we already know that $u \in W^{2,2}$, partial integration would yield an equation for $w = u_{,j}$ namely

$$-\int_\Omega \frac{\partial A^i}{\partial p_k} w_{,k}\varphi_{,i} + \frac{\partial A^i}{\partial z} w\varphi_i + \frac{\partial A^i}{\partial x^j}\varphi_{,i} = -\int_\Omega f_{,j}\varphi$$

and thus

$$-\left(\frac{\partial A^i}{\partial p_k} w_{,k} + \frac{\partial A^i}{\partial z} w + \frac{\partial A^i}{\partial x^j}\right)_{,i} = f_{,j},$$

which formally is a divergence form equation as treated above. We would like to apply the $W^{2,2}$-estimate to deduce that $w \in W^{2,2}$ and in turn $u \in W^{3,2}$. However, the x dependence of the operator on the left hand side is now hidden in the coefficients

$$\frac{\partial A^i}{\partial p_k} = \frac{\partial A^i}{\partial p_k}(Du(x), u(x), x)$$

and similarly for the other coefficients.[5] However, the assumptions of Theorem 4.2.1 are not met, since we do not know that these coefficients are differentiable with respect to x. To get higher regularity, more sophisticated techniques are necessary and we will not perform this here. Instead we restrict to higher regularity for linear equations.

4.2.3 Theorem (Higher interior estimates)**.** *Let $n, m \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ open and $f \in W^{m,2}_{\mathrm{loc}}(\Omega)$. For $1 \le i, j \le n$ let $a^{ij}, a^i \in C^{m+1}(\bar{\Omega})$ and $b^i, d \in C^m(\bar{\Omega})$ and suppose (a^{ij}) is strictly positive definite with ellipticity constant λ_0. Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a weak solution of*

$$-(a^{ij}u_{,j} + a^i u)_{,i} + b^i u_{,i} + du = f,$$

then

$$u \in W^{m+2,2}_{\mathrm{loc}}(\Omega)$$

and for all $\Omega' \Subset \Omega'' \Subset \Omega$ there exists $c > 0$ such that

$$\|u\|_{m+2,\Omega'} \le c(\|f\|_{m,2,\Omega''} + \|u\|_{1,2,\Omega''}),$$

where c depends on λ_0, $|a^{ij}|_{m+1,\Omega}, |a^i|_{m+1,\Omega}, |b^i|_{m+1,\Omega}, |d|_{m+1,\Omega}$, $\mathcal{L}^n(\Omega)$ and on $\mathrm{dist}(\Omega', \partial\Omega'')$.

Proof. By induction. For $m = 0$ this is Theorem 4.2.1, since

$$A(p,z,x) = a^{ij}(x)p_j + a^i(x)z, \quad B(p,z,x) = b^i p_i + dz$$

and

$$|\partial_{x^k} A(p,z,x)| = |a^{ij}_{,k}p_j + a^i_{,k}z| \le c^1_A(|p| + |z|),$$

$$|\partial_z A| + |\partial_{p^j} A| = |a^i| + |a^{ij}| \le c^1_A$$

[5] Note that u not the unknown function anymore and fixed.

and a^{ij} is positive definite. Similar estimates hold for B. Also note that $c_A^2 = c_B^2 = 0$. Now let $m > 0$ and suppose the claim holds for $m-1$. First of all $u \in W^{m+1,2}_{\mathrm{loc}}(\Omega)$ with the corresponding estimate. For $1 \leq k \leq n$ choose

$$\eta = \varphi_{,k}$$

with $\varphi \in C_c^\infty(\Omega)$. Then

$$\int_\Omega (a^{ij}u_{,j} + a^i u)\eta_{,i} + \int_\Omega (b^i u_{,i} + du)\eta = \int_\Omega f\eta$$

and hence, with $w = u_{,k}$,

$$\begin{aligned} &\int_\Omega (a^{ij}_{,k}u_{,j} + a^{ij}w_{,j} + a^i_{,k}u + a^i w)\varphi_{,i} + \int_\Omega (b^i_{,k}u_{,i} + b^i w_{,i} + d_{,k}u + dw)\varphi \\ = &\int_\Omega f_{,k}\varphi. \end{aligned}$$

Thus $w = u_{,k} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ satisfies

$$\begin{aligned} -(a^{ij}w_{,j} + a^i w)_{,i} + b^i w_{,i} + dw = f_{,k} + (a^{ij}_{,k}u_{,j})_{,i} + (a^i_{,k}u)_{,i} - b^i_{,k}u_i - d_{,k}u \\ =: F \in W^{m-1,2}_{\mathrm{loc}}(\Omega). \end{aligned}$$

Hence by induction hypothesis $w \in W^{m+1,2}_{\mathrm{loc}}(\Omega)$. Let $\Omega' \Subset \Omega''' \Subset \Omega''$. Then

$$\|w\|_{m+1,2,\Omega'} \leq c(\|F\|_{m-1,2,\Omega'''} + \|w\|_{1,2,\Omega'''}).$$

There hold

$$\|w\|_{1,2,\Omega'''} \leq \|u\|_{2,2,\Omega'''} \leq c(\|f\|_{2,\Omega''} + \|u\|_{1,2,\Omega''})$$

and

$$\|F\|_{m-1,2,\Omega'''} \leq c(\|f\|_{m,2,\Omega''} + \|u\|_{m+1,2,\Omega'''}) \leq c(\|f\|_{m,2,\Omega''} + \|u\|_{1,2,\Omega''}).$$

Combining these estimates gives the claim. □

Due to Exercise 3.4.4 we obtain the following local regularity result.

4.2.4 Corollary. *Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ open. Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a distributional solution to the linear problem*

$$-(a^{ij}u_{,j} + a^i u)_{,i} + b^i u_{,i} + du = f,$$

where (a^{ij}) is locally strictly positive definite and f as well as all coefficients are smooth. Then $u \in C^\infty(\Omega)$.

Proof. We know that $u \in W^{m,2}_{\mathrm{loc}}(\Omega)$ for all $m \in \mathbb{N}$. Let $\Omega' \Subset \Omega$ and choose a cut-off function $\eta \in C_c^\infty(\Omega)$ with

$$\eta_{|\bar{\Omega}'} = 1.$$

Then $\eta u \in W^{m,2}_0(\Omega)$ for all m and hence $u \in C^\infty(\Omega')$. □

Boundary estimates

In this section we extend the previous regularity results to the boundary $\partial\Omega$. Roughly, on a domain Ω with smooth boundary and with data smooth up to the boundary, we want to conclude that a $W_0^{1,2}(\Omega)$ solution is of class $C^\infty(\bar{\Omega})$. We proceed as in the previous subsection, proving the first step for general divergence form operators and the inductive step for linear operators.

In order to prove boundary estimates, we have to transform the equation onto a simpler domain. We use straightening of the boundary. Hence we first have to calculate, how an equation in divergence form transforms under a change of coordinates.

4.2.5 Lemma. *Let $n \in \mathbb{N}$, $\Omega, \tilde{\Omega} \subset \mathbb{R}^n$ open and $\psi \in C^1(\Omega, \tilde{\Omega})$ be a coordinate transformation. Let $f \in L^1_{\text{loc}}(\Omega)$ and $A, B \in \mathbb{R}^{\mathbb{R}^n \times \mathbb{R} \times \Omega}$. Let $u \in W^{1,2}_{\text{loc}}(\Omega)$ be a weak solution of*

$$-\operatorname{div} A(Du, u, \cdot) + B(Du, u, \cdot) = f.$$

Then $\tilde{u} = u \circ \psi^{-1} \in W^{1,2}_{\text{loc}}(\tilde{\Omega})$ is a weak solution of

$$\begin{aligned} & -\operatorname{div}\left(\sqrt{\det g}(D\psi(\cdot) \circ A)(D\tilde{u}(\cdot) D\psi \circ \psi^{-1}, \tilde{u}, \cdot)\right) \\ = \ & (f - B(D\tilde{u}(\cdot) D\psi \circ \psi^{-1}, \tilde{u}, \cdot))\sqrt{\det g}, \end{aligned}$$

where g is the Gramian matrix associated to ψ.

Proof. Let $\tilde{\varphi} \in C_c^\infty(\tilde{\Omega})$ and $\varphi = \tilde{\varphi} \circ \psi$. Then

$$\begin{aligned} & \int_{\tilde{\Omega}} \tilde{\varphi}_{,k}(\psi^k_{,i}(\tilde{x}) \circ A^i)(D\tilde{u}(\tilde{x}) D\psi \circ \psi^{-1}(\tilde{x}), \tilde{u}(\tilde{x}), \tilde{x})\sqrt{\det g(\tilde{x})}\ d\tilde{x} \\ = & \int_{\tilde{\Omega}} \varphi_{,i}(\psi^{-1}(\tilde{x})) A^i(Du(\psi^{-1}(\tilde{x})), u \circ \psi^{-1}(\tilde{x}), \tilde{x})\sqrt{\det g(\tilde{x})}\ d\tilde{x} \\ = & \int_{\Omega} \varphi_{,i}(x) A^i(Du(x), u(x), x)\ dx \\ = & \int_{\Omega} (f(x) - B(Du, u, x))\varphi(x)\ dx \\ = & \int_{\tilde{\Omega}} (f(\psi^{-1}(\tilde{x})) - B(D\tilde{u}(\tilde{x}) D\psi \circ \psi^{-1}(\tilde{x}), \tilde{u}(\tilde{x}), \tilde{x}))\varphi(\tilde{x})\sqrt{\det g(\tilde{x})}\ d\tilde{x}. \end{aligned}$$

□

Due to this lemma it will be possible to reduce the boundary estimates to the canonical situation of a straight boundary; in the sequel we use the notation

$$B^+_\rho(0) = B_\rho(0) \cap \{x^n > 0\}, \quad \rho > 0,$$

and $\bar{B}^+_\rho(0)$ for its closure. After the transformation we will not be in the situation that the transformed solution u is in $W_0^{1,2}(B^+_\rho(0))$, but it will only be zero on the flat boundary portion of $B^+_\rho(0)$. Hence we have to define what this is supposed to mean.

4.2.6 Definition (Weak boundary values)**.** Let $n, m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open and $T \subset \partial\Omega$ closed. We say that a function $u \in W^{m,2}(\Omega)$ *equals* $\psi \in W^{m,2}(\Omega)$ *on T in the sense of* $W^{m,2}$,

$$u_{|T} = \psi$$

if $u-\psi$ can be approximated in the $W^{m,2}$-norm by functions $\varphi \in C_c^\infty(\bar\Omega \backslash T)$.

4.2.7 Theorem (Local $W^{2,2}$-boundary estimates). *Let $n \in \mathbb{N}$, $0<\rho\le 1$ and $f \in L^2(B_\rho^+(0))$. Let $A \in C^1(\mathbb{R}^n \times \mathbb{R} \times B_\rho^+(0), \mathbb{R}^n)$ satisfy*

$$|\partial_x A(p,z,x)| \le c_A^1(c_A^2 + |p| + |z|), \quad |\partial_z A| + |\partial_p A| \le c_A^1$$

for some $c_A^i > 0$. Suppose $B \in \mathbb{R}^{\mathbb{R}^n \times \mathbb{R} \times B_\rho^+(0)}$ is measurable and satisfies for almost every (p,z,x),

$$|B(p,z,x)| \le c_B^1(c_B^2 + |p| + |z|).$$

Let

$$\mathcal{L}_{A,B} \colon W^{1,2}(B_\rho^+(0)) \to \mathcal{D}'(B_\rho^+(0))$$

be a divergence form partial differential operator of second order in $B_\rho^+(0)$,

$$\mathcal{L}_{A,B}(u) = -\operatorname{div} A(Du,u,\cdot) + B(Du,u,\cdot),$$

which is strictly elliptic in $W^{1,2}(B_\rho^+(0))$ with ellipticity constant $\lambda > 0$. Let $u \in W^{1,2}(B_\rho^+(0))$ be a solution of the problem

$$\begin{aligned} \mathcal{L}_{A,B}(u) &= f \\ u_{|\{x^n=0\}} &= 0 \quad \text{in } W^{1,2}. \end{aligned}$$

Then for all $0<\rho_1<\rho$ there holds

$$u \in W^{2,2}(B_{\rho_1}^+(0))$$

and there exists c depending on n, c_A^1, c_B^1, λ, and $\rho-\rho_1$ such that

$$\|u\|_{2,2,B_{\rho_1}^+(0)} \le c(c_A^2 + c_B^2 + \|u\|_{1,2,B_\rho^+(0)} + \|f\|_{2,B_\rho^+(0)}).$$

Proof. For $\rho_1 < \rho_2 < \rho$ choose

$$\eta \in C_c^\infty(B_{\rho_2}(0)), \quad \eta_{|B_{\rho_1}(0)} \equiv 1.$$

We first estimate the difference quotient in horizontal directions, i.e. for sufficiently small h we choose the test function

$$-\Delta_{-h}^k(\eta^2 \Delta_h^k u) \in W_0^{1,2}(B_\rho^+(0)),^{6} \quad 1 \le k \le n-1.$$

Then

$$\int_{B_{\rho_2}^+(0)} \Delta_h^k A^i (\Delta_h^k u \eta^2)_{,i} = -\int_{B_{\rho_2}^+(0)} (f-B)\Delta_{-h}^k(\Delta_h^k u\eta^2).$$

As in the proof of Theorem 4.2.1 we write for $0 < h < \operatorname{dist}(B_{\rho_2}^+(0), \partial B_\rho)$

$$\begin{aligned} &\Delta_h^k A^i(Du,u,\cdot) \\ = {} & \frac{1}{h}\int_0^1 \frac{d}{dt} A^i(tDu(\cdot+he_k) + (1-t)Du, tu(\cdot+he_k) + (1-t)u, \cdot + the_k)\, dt \\ = {} & A^{ij}(\Delta_h^k u)_{,j} + a^i \Delta_h^k u + \alpha_k^i, \end{aligned}$$

[6] We leave this as an exercise.

where

$$A^{ij} = \int_0^1 \frac{\partial A^i}{\partial p_j}(tDu(\cdot + h) + (1-t)Du, tu(\cdot + h) + (1-t)u, \cdot + th)\ dt \quad (4.9)$$

and

$$a^i = \int_0^1 \frac{\partial A^i}{\partial z}, \quad \alpha_k^i = \int_0^1 \frac{\partial A^i}{\partial x^k},$$

where the integrand terms are also evaluated at the convex combinations as in (4.9). We obtain

$$\begin{aligned}
&\int_{B_{\rho_2}^+(0)} \eta^2 A^{ij}(\Delta_h^k u)_{,i}(\Delta_h^k u)_{,j} \\
= & -\int_{B_{\rho_2}^+(0)} 2\eta\eta_{,i} A^{ij}(\Delta_h^k u)_{,j}\Delta_h^k u - \int_{B_{\rho_2}^+(0)} a^i \Delta_h^k u(\eta^2(\Delta_h^k u)_{,i} + 2\eta\eta_{,i}\Delta_h^k u) \\
& -\int_{B_{\rho_2}^+(0)} \alpha_k^i(\eta^2(\Delta_h^k u)_{,i} + 2\eta\eta_{,i}\Delta_h^k u) - \int_{B_{\rho_2}^+(0)} (f-B)\Delta_{-h}^k(\eta^2\Delta_h^k u) \\
\le & \int_{B_{\rho_2}^+(0)} \frac{\eta^2}{2} A^{ij}(\Delta_h^k u)_{,i}(\Delta_h^k u)_{,j} + 2\int_{B_{\rho_2}^+(0)} A^{ij}\eta_{,i}\eta_{,j}(\Delta_h^k u)^2 \\
& + \frac{c\epsilon}{2}\int_{B_{\rho_2}^+(0)} \eta^2|D(\Delta_h^k u)|^2 + \frac{c}{2\epsilon}\int_{B_{\rho_2}^+(0)} \eta(c_A^2 + |u|^2 + |Du|^2 + (\Delta_h^k u)^2) \\
& + \frac{\epsilon}{2}\int_{B_{\rho_2}^+(0)} |D(\eta^2\Delta_h^k u)|^2 + \frac{1}{2\epsilon}\int_{B_{\rho_2}^+(0)} |f|^2 + \frac{1}{2\epsilon}\int_{B_{\rho_2}^+(0)} |B(Du, u, \cdot)|^2.
\end{aligned}$$

As in the proof of Theorem 4.2.1 this implies

$$\|D(\Delta_h^k u)\|_{2,B_{\rho_1}^+(0)} \le c(c_A^2 + c_B^2 + \|u\|_{1,2,B_{\rho_2}^+(0)} + \|f\|_{2,B_{\rho_2}^+(0)}).$$

From Lemma 3.3.14 we obtain that Du is weakly differentiable in any direction $1 \le k \le n-1$ and

$$\sum_{i+j<2n} \|u_{,ij}\|_{2,B_{\rho_1}^+(0)} \le c(c_A^2 + c_B^2 + \|u\|_{1,2,B_{\rho_2}^+(0)} + \|f\|_{2,B_{\rho_2}^+(0)}).$$

To estimate $u_{,nn}$ we use the differential equation directly. Since we already know from Theorem 4.2.1 that $u \in W_{\text{loc}}^{2,2}(B_{\rho_2}^+(0))$ and A is differentiable with bounded $\partial_p A$ and $\partial_z A$, from the chain rule we obtain

$$-\frac{\partial A^i}{\partial p_j}(Du, u, \cdot)u_{,ij} - \frac{\partial A^i}{\partial x^i}(Du, u, \cdot) - \frac{\partial A^i}{\partial z}(Du, u, \cdot)u_i = f - B(Du, u, \cdot).$$

Using $a^{nn} \ge \lambda$, we obtain almost everywhere

$$|u_{,nn}| \le c\sum_{i+j<2n} |u_{,ij}| + c_A^1(c_A^2 + |u| + |Du|) + |f| + c_B^1(c_B^2 + |u| + |Du|).$$

□

Similar to the interior estimates we prove higher boundary regularity. Therefore we need the following lemma:

4.2.8 Lemma. *Let $n \in \mathbb{N}$, $0 < \rho_2 < \rho$ and $u \in W^{2,2}(B^+_\rho(0))$ vanish on $\{x^n = 0\}$ in the sense of $W^{1,2}$. Then for $1 \le k \le n-1$, $u_{,k} \in W^{1,2}(B^+_{\rho_2}(0))$ vanishes on $\{x^n = 0\}$ in the sense of $W^{1,2}$.*

Proof. Let $\eta \in C^\infty_c(B_\rho(0))$ with $\eta_{|B^+_{\rho_2}(0)} = 1$, then $\eta u \in W^{2,2}(B^+_\rho(0))$. For $\rho_2 < \rho_1 < \rho$ and small h, the difference quotients

$$\Delta^k_h(\eta u) \in W^{1,2}_0(B^+_{\rho_1}(0))$$

converge to $(\eta u)_{,k}$ in $W^{1,2}_0(B^+_{\rho_1}(0))$ as $h \to 0$, Lemma 3.3.13.[7] Due to the closedness of $W^{1,2}_0(B^+_{\rho_1}(0))$ we have

$$(\eta u)_{,k} \in W^{1,2}_0(B^+_{\rho_1}(0)).$$

Hence $(\eta u)_{,k}$ vanishes on $\{x^n = 0\} \cap B^+_{\rho_1}(0)$ and hence the claimed result follows. □

4.2.9 Theorem. *Let $n, m \in \mathbb{N}$, $0 < \rho \le 1$ and $f \in W^{m,2}(B^+_\rho(0))$. For $1 \le i, j \le n$ let $a^{ij}, a^i \in C^{m+1}(\bar{B}^+_\rho(0))$ and $b^i, d \in C^m(\bar{B}^+_\rho(0))$ and suppose (a^{ij}) is strictly positive definite with ellipticity constant λ. Let $u \in W^{1,2}(B^+_\rho(0))$ be a weak solution of*

$$\begin{aligned} -(a^{ij}u_{,j} + a^i u)_{,i} + b^i u_{,i} + du &= f \\ u_{|\{x^n=0\}} &= 0 \quad \text{in } W^{1,2}. \end{aligned}$$

Then for all $0 < \rho_1 < \rho$ there holds

$$u \in W^{m+2,2}(B^+_{\rho_1}(0))$$

and

$$\|u\|_{m+2,2,B^+_{\rho_1}(0)} \le c\left(\|f\|_{m,2,B^+_\rho(0)} + \|u\|_{1,2,B^+_\rho(0)}\right),$$

where c depends on λ, $|a^{ij}|_{m+1,\Omega}, |a^i|_{m+1,\Omega}, |b^i|_{m+1,\Omega}, |d|_{m+1,\Omega}$ and on $\rho - \rho_1$.

Proof. For $m = 0$ this is Theorem 4.2.7, which implies

$$\|u\|_{2,2,B^+_{\rho_1}(0)} \le c\left(\|u\|_{1,2,B^+_\rho(0)} + \|f\|_{2,B^+_\rho(0)}\right),$$

Let $m > 0$ and suppose the result is true for $m - 1$. Let $\rho_1 < \rho_2 < \rho$. First of all

$$u \in W^{m+1,2}(B^+_{\rho_2}(0))$$

with the corresponding estimate. Then, for $1 \le k \le n-1$,

$$w = u_{,k} \in W^{m,2}(B^+_{\rho_2}(0)),$$

[7] Note that the formal assumptions of this lemma are not quite met, since $\text{dist}(B^+_{\rho_2}(0), \partial B^+_\rho(0)) = 0$. But due to $1 \le k \le n-1$ we do not leave the domain of definition of u and hence the proof of Lemma 3.3.13 carries over.

from Lemma 4.2.8 we obtain $w = 0$ on $\{x^n = 0\}$ in the sense of $W^{1,2}$ on a possibly slightly smaller set and w satisfies

$$-(a^{ij}w_{,j} + a^i w)_{,i} + b^i w_{,i} + dw = f_{,k} + (a^{ij}_{,k}u_{,j})_{,i} + (a^i_{,k}u)_{,i} - b^i_{,k}u_i - d_{,k}u$$
$$=: F \in W^{m-1,2}(B^+_{\rho_2}(0)).$$

By induction hypothesis there holds

$$w \in W^{m+1,2}(B^+_{\rho_1}(0))$$

and

$$\|w\|_{m+1,2,B^+_{\rho_1}(0)} \leq c\left(\|F\|_{m-1,2,B^+_{\rho_2}(0)} + \|w\|_{1,2,B^+_{\rho_2}(0)}\right)$$

and the proof can be completed as in Theorem 4.2.3. □

Combining all of the interior and boundary estimates by using a partition of unity, we obtain the full $W^{m,2}$-existence and regularity theorem for weak solutions, for which we can also include more general boundary values. This is the main result of this chapter and the exact proof is recommended as an exercise.

4.2.10 Theorem. *Let $n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ open with C^{m+2}-boundary, $f \in W^{m,2}(\Omega)$ and $\psi \in W^{m+2,2}(\Omega)$. Let $a^{ij}, a^i \in C^{m+1}(\bar{\Omega})$, $b^i, d \in C^m(\bar{\Omega})$ and, for some $\lambda > 0$*

$$\forall (\xi_i) \in \mathbb{R}^n \colon a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2.$$

Let $u \in W^{1,2}(\Omega)$ be a weak solution of the equation

$$\begin{aligned} (a^{ij}u_{,j} + a^i u)_{,i} + b^i u_{,i} + du &= f \\ u_{|\partial\Omega} &= \psi \quad \text{in } W^{1,2}. \end{aligned} \tag{4.10}$$

Then

$$u \in W^{m+2,2}(\Omega)$$

and there holds

$$\|u\|_{m+2,2,\Omega} \leq c(\|u\|_{2,\Omega} + \|f\|_{m,2,\Omega} + \|\psi\|_{m+2,2,\Omega}),$$

where c only depends on the data of the problem.[8] *If in addition there holds*

$$d + a^i_{,i} \leq 0,$$

then (4.10) admits a unique solution in $W^{m+2,2}(\Omega)$.

This theorem, together with Corollary 3.5.4, implies that if all data are smooth, the solution is of class $C^\infty(\bar{\Omega})$. The only piece that is missing for the classical Dirichlet problem to be solved, is that we have to show that

$$v \in C^\infty(\bar{\Omega}) \cap W^{1,2}_0(\Omega) \quad \Rightarrow \quad v_{|\partial\Omega} = 0.$$

---

[8] and not on u.

4.2.11 Lemma. *Let $n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ open with C^1-boundary. Let $v \in C^1(\bar{\Omega})$. Then*

$$v \in W_0^{1,2}(\Omega) \quad \Rightarrow \quad v_{|\partial\Omega} = 0.$$

Proof. We prove that the restriction operator

$$R\colon (C^1(\bar{\Omega}), \|\cdot\|_{1,2,\Omega}) \to L^1(\partial\Omega)$$
$$u \mapsto u_{|\partial\Omega}$$

is continuous. There suppose first, that $\operatorname{supp}(u) \subset U$, where U is the domain of a straightening function ψ. Then

$$\begin{aligned}\int_{\partial\Omega} |u| &= \int_{\psi(\partial\Omega\cap U)} |u \circ \psi^{-1}(\hat{x}, 0)| \sqrt{\det g_{\partial\Omega}(\hat{x})}\ d\hat{x} \\ &\leq c \int_{\psi(\partial\Omega\cap U)} \int_0^\infty |Du(\psi^{-1}(\hat{x}, x^n))|\ d\hat{x} dx^n \\ &\leq c \int_\Omega |Du|.\end{aligned}$$

A partition of unity gives the continuity of R. Hence there is a unique extension of R to $W^{1,2}(\Omega)$. If $v \in W_0^{1,2}(\Omega)$, then a sequence $(\varphi_k)_{k\in\mathbb{N}}$ of functions in $C_c^\infty(\Omega)$ converges to v in the $W^{1,2}$-norm and hence

$$0 = R(\varphi_k) \to R(v) = v_{|\partial\Omega}.$$

□

We obtain existence and regularity of solutions to the classical Dirichlet problem.

4.2.12 Theorem. *Let $n \in \mathbb{N}$ and $\Omega \Subset \mathbb{R}^n$ open with smooth boundary. Let $u \in W^{1,2}(\Omega)$ be a distributional solution to the linear Dirichlet problem*

$$\begin{aligned}(a^{ij}u_{,j} + a^i u)_{,i} + b^i u_{,i} + du &= f \\ u_{|\partial\Omega} &= \psi \quad \text{in } W^{1,2},\end{aligned}$$

where (a^{ij}) is strictly positive definite and f, ψ as well as all coefficients are smooth up to the boundary. Then $u \in C^\infty(\bar{\Omega})$ and

$$u_{|\partial\Omega} = \psi_{|\partial\Omega}.$$

If in addition there holds

$$d + a^i_{,i} \leq 0,$$

then the classical Dirichlet problem

$$\begin{aligned}(a^{ij}u_{,j} + a^i u)_{,i} + b^i u_{,i} + du &= f \quad \text{in } \Omega \\ u_{|\partial\Omega} &= \psi \quad \text{on } \partial\Omega\end{aligned}$$

is uniquely solvable in $C^\infty(\bar{\Omega})$.

4.3 Dirichlet spectrum of the Laplace operator

From Theorem 3.1.21 we obtain the following spectral theorem for the Laplace operator.

4.3.1 Theorem. *Let $n \in \mathbb{N}$ and $\Omega \Subset \mathbb{R}^n$ with smooth boundary. Then*

$$-\Delta \colon W_0^{1,2} \to \mathcal{D}'(\Omega)$$

has countably many eigenvalues λ, i.e. exists $u \neq 0$ such that

$$\int_\Omega \langle Du, D\varphi \rangle = \lambda \int_\Omega u\varphi \quad \forall \varphi \in C_c^\infty(\Omega).$$

If we order the eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \dots,$$

then

$$\lim_{i \to \infty} \lambda_i = \infty.$$

The normalised eigenfunctions u_i are of the class $C^\infty(\bar{\Omega})$ and form an L^2-orthonormal basis of $L^2(\Omega)$. Furthermore there holds

$$\int_\Omega \langle Du_i, Du_j \rangle = \lambda_i \delta_{ij} \quad \forall 1 \leq i, j < \infty.$$

Proof. We justify the applicability of Theorem 3.1.21. Set $H = W_0^{1,2}(\Omega)$,

$$K(u, v) = \int_\Omega uv, \quad B(u, v) = \int_\Omega \langle Du, Dv \rangle .$$

Then both K and B are symmetric and continuous due to Hölder's inequality. Furthermore K is compact, since for a bounded sequence $(u_k)_{k \in \mathbb{N}}$ in $W_0^{1,2}(\Omega)$, Rellich's theorem implies the existence of a convergent subsequence

$$u_k \to u$$

in $L^2(\Omega)$. B is coercive relative K, since

$$B(u, u) = \int_\Omega |Du|^2 = \|u\|_{1,2,\Omega}^2 - \|u\|_{2,\Omega}^2 = \|u\|_{1,2,\Omega} - K(u, u).$$

Thus from Theorem 3.1.21 we obtain a countable family of eigenvalues λ_i and eigenfunctions u_i, which are smooth due to the regularity theorem Theorem 4.2.12. They are complete in $W_0^{1,2}$, but since the L^2-closure of $W_0^{1,2}(\Omega)$ equals $L^2(\Omega)$, they also form a basis of $L^2(\Omega)$. The orthogonality relations follow as well. ∎

Chapter 5

The model equations

In this chapter we collect some classical results for the model equations, namely for the Laplace- and Poisson equation, the heat equation and the wave equation. We will discuss fundamental solutions and give some existence result for the heat equation using Laplace eigenfunctions.

5.1 Laplace equation

Mean value property and its consequences

The following theorem is the well known mean value property for harmonic functions, which was first proved by Riemann in the case $n = 2$ for harmonic functions, [15]. The presentation of most of the results in this section is taken from [5].

5.1.1 Theorem. *Let $2 \leq n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ open. For a function $u \in L^1_{\text{loc}}(\Omega)$ the following statemants are equivalent.*

(i) $u \in C^\infty(\Omega)$ and $\Delta u = 0$.

(ii) u satisfies the mean value property, i.e. for almost every $x \in \Omega$ there holds

$$u(x) = \frac{1}{\omega_n r^n} \int_{B_r(x)} u \quad \forall B_r(x) \Subset \Omega,$$

where ω_n is the measure of the n-dimensional unit ball.

Proof. (i) $\Rightarrow$ (ii): Define

$$\begin{aligned} f(r) = \frac{1}{\omega_n r^n} \int_{B_r(x)} u &= \frac{1}{\omega_n} \int_{B_1(0)} u(x + rz)\, dz \\ &= \frac{1}{\omega_n} \int_0^1 \int_{\mathbb{S}^{n-1}} u(x + rs\xi) s^{n-1} d\xi ds. \end{aligned}$$

Differentiation gives

$$f'(r) = \frac{1}{\omega_n} \int_0^1 \int_{\mathbb{S}^{n-1}} s^{n-1} \langle \nabla u(x + rs\xi), s\xi \rangle \, d\xi ds.$$

Then, letting $v(z) = u(x + rsz)$, we obtain

$$
\begin{aligned}
f'(r) &= \frac{1}{\omega_n} \int_0^1 \frac{s^{n-1}}{r} \int_{\mathbb{S}^{m-1}} \langle \nabla v(\xi), \xi \rangle \, d\xi ds \\
&= \frac{1}{\omega_n} \int_0^1 \frac{s^{n-1}}{r} \int_{B_1(0)} \Delta v \ ds \\
&= 0.
\end{aligned}
$$

Thus, f is constant and

$$
|f - u(x)| \leq \frac{1}{\omega_n r^n} \int_{B_r(x)} |u - u(x)| \leq \sup_{B_r(x)} |u - u(x)| \to 0, \quad r \to 0,
$$

due to continuity.

(ii) $\Rightarrow$ (i): First we note that a function $u \in L^1_{\text{loc}}(\Omega)$ satisfying the mean value property is continuous in Ω, since for $x, y \in \Omega$ and small $\epsilon > 0$ we have

$$
\begin{aligned}
|u(x) - u(y)| &\leq \frac{1}{\omega_n \epsilon^n} \left| \int_{B_\epsilon(x)} u - \int_{B_\epsilon(y)} u \right| \\
&\leq \frac{1}{\omega_n \epsilon^n} \left(\int_{B_\epsilon(x) \backslash B_\epsilon(y)} |u| + \int_{B_\epsilon(y) \backslash B_\epsilon(x)} |u| \right) \\
&\to 0, \quad \text{if } y \to x.
\end{aligned}
$$

Thus the mean value property is actually satisfied everywhere. Now we calculate the convolution with a radially symmetric mollifier:

$$
\begin{aligned}
u_\epsilon(x) &= \int_{B_\epsilon(x)} u(y) \eta_\epsilon(|y - x|) \ dy = \int_0^\epsilon \int_{\partial B_1(x)} u(r\xi) \eta_\epsilon(r) r^{n-1} \ d\xi dr \\
&= \int_0^\epsilon \eta_\epsilon(r) r^{n-1} \int_{\partial B_1(x)} u(r\xi) \ d\xi dr \\
&= \int_0^\epsilon \eta_\epsilon(r) r^{n-1} \left(r^{1-n} \frac{d}{dr} \int_{B_r(x)} u \right) dr \\
&= \int_0^\epsilon \eta_\epsilon(r) \frac{d}{dr} (\omega_n r^n u(x)) dr \\
&= u(x) \int_0^\epsilon n \omega_n r^{n-1} \eta_\epsilon(r) dr = u(x).
\end{aligned}
$$

Thus u coincides with its convolution and is consequently smooth. Suppose at some $x \in \Omega$ we had $\Delta u(x) > 0$. Then the function f from the first part of the proof would be strictly increasing, in contradiction to the mean value property. Similarly $\Delta u(x) < 0$ leads to a contradiction. □

For harmonic functions we obtain nice derivative estimates. For an explicit value of the constant involved see [5, Thm. 2.10].

5.1.2 Proposition. *Let $2 \leq n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open and $u \in C^\infty(\Omega)$ harmonic. Then for $B_r = B_r(x) \subset \Omega$ there holds*

$$
|u_{,\alpha}(x)| \leq C(n, |\alpha|) \left(\frac{1}{r} \right)^{|\alpha|} \sup_{B_r(x)} |u|.
$$

Proof. By induction on the order of the multiindex.

$|\alpha| = 1$: The function $u_{,i}$ is harmonic in Ω and satisfies

$$u_{,i}(x) = \frac{2^n}{\omega_n r^n} \int_{B_{\frac{r}{2}}} u_{,i} = \frac{2^n}{\omega_n r^n} \int_{\partial B_{\frac{r}{2}}} u \langle \nu, e_i \rangle .$$

Thus

$$|u_{,i}(x)| \leq \frac{2n}{r} \sup_{B_{\frac{r}{2}}} |u|.$$

Let the claim hold for $|\alpha| = k$, then

$$|u_{,i_1 \dots i_{k+1}}(x)| \leq \frac{2n}{r} \sup_{B_{\frac{r}{2}}} |u_{,i_1 \dots i_k}| \leq \frac{2n}{r^{k+1}} C(m,k) \sup_{B_r} |u|.$$

□

The following *Liouville theorem* is a direct consequence.

5.1.3 Theorem. *Every bounded and harmonic function on $\mathbb{R}^n$, $n \geq 1$, is constant.*

Proof. For $n = 1$ every harmonic function is linear, so in this case the result holds. For $n \geq 2$, in Proposition 5.1.2 let $r \to \infty$. □

There is another corollary, the proof of which is an exercise:

5.1.4 Exercise. Let $n \geq 1$. Then every bounded sequence of harmonic functions on a domain $\Omega \subset \mathbb{R}^n$ contains a subsequence, which converges locally uniformly to a harmonic function on Ω.

From the mean value property we obtain the following famous inequality.

5.1.5 Theorem (Harnack). *Let $2 \leq n \in \mathbb{N}$ and u be a non-negative harmonic function on an open set $\Omega \subset \mathbb{R}^n$. Then for any connected $\Omega' \Subset \Omega$ there exists a constant $c = c(n, \Omega', \Omega)$, such that*

$$\sup_{\Omega'} u \leq c \inf_{\Omega'} u.$$

Proof. Let $y \in \Omega$ and $B_{4R}(y) \subset \Omega$. Then for $x_1, x_2 \in B_R(y)$ there holds

$$u(x_1) = \frac{1}{\omega_n R^n} \int_{B_R(x_1)} u \leq \frac{1}{\omega_n R^n} \int_{B_{2R}(y)} u,$$

$$u(x_2) = \frac{1}{\omega_n 3^n R^n} \int_{B_{3R}(x_2)} u \geq \frac{1}{\omega_n 3^n R^n} \int_{B_{2R}(y)} u$$

and hence

$$u(x_1) \leq 3^n u(x_2)$$

and thus

$$\sup_{B_R(y)} u \leq 3^n \inf_{B_R(y)} u.$$

Let C be a closed path between two points x and z and cover C by finitely many balls of radius R, such that $4R < \operatorname{dist}(C, \partial\Omega)$. Apply the previous estimate to each ball. □

Fundamental solution

We want to find a nontrivial radially symmetric solution Γ of the Laplace equation in $\mathbb{R}^n$. Therefore we calculate $\Delta\Gamma$, where

$$\Gamma(x) = \gamma(r) = \gamma(|x|).$$

There holds

$$\Gamma_{,i} = \gamma' \frac{x_i}{|x|}, \quad \Gamma_{,ij} = \gamma'' \frac{x_i x_j}{|x|^2} + \gamma' \frac{\delta_{ij}}{|x|} - \gamma' \frac{x_i x_j}{|x|^3}.$$

Thus

$$\Delta\Gamma = \gamma'' + \frac{n-1}{r}\gamma'.$$

Hence if we put

$$\gamma(r) = \begin{cases} \log r, & n = 2 \\ r^{2-n}, & n \geq 3 \end{cases}$$

for $r > 0$, we may make the following definition.

5.1.6 Definition (Fundamental solution of Laplace's equation)**.** Let $2 \leq n \in \mathbb{N}$. The function

$$\Gamma \colon \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$$
$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} \log(|x|), & n = 2 \\ \frac{1}{n(2-n)\omega_n} |x|^{2-n}, & n \geq 3 \end{cases}$$

is called the *fundamental solution of the Laplace equation.*

Thus for every $y \in \mathbb{R}^n$, the function $\Gamma(\cdot - y)$ is harmonic in $\mathbb{R}^n \backslash \{y\}$. But we can say even more.

5.1.7 Proposition. *Let $2 \leq n \in \mathbb{N}$ and $y \in \mathbb{R}^n$ then $\Gamma(\cdot - y) \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$ and*

$$\Delta\Gamma(\cdot - y) = \delta_y.$$[1]

Proof. To show that $\Gamma(\cdot - y) \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$ we note that there holds

$$\partial_{x^i}\Gamma(x - y) = \frac{1}{n\omega_n} \frac{x_i - y_i}{|x - y|^n}$$

and hence we obtain the following estimate:

$$|D_x\Gamma(x - y)| \leq \frac{1}{n\omega_n} |x - y|^{1-n}.$$

[1] This is the reason for the choice of the constants in the definition of the fundamental solution.

Hence the weak derivative, if it exists, will be integrable, as well as the function itself. Thus it remains to check that $\Gamma(\cdot - y)$ is weakly differentiable. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then for $\epsilon > 0$ there holds

$$\begin{aligned}\int_{\mathbb{R}^n} \Gamma(\cdot - y)\varphi_{,i}\ dx &= \int_{\mathbb{R}^n\setminus \bar{B}_\epsilon(y)} \Gamma(x-y)\varphi_{,i}(x)\ dx + \int_{\bar{B}_\epsilon(y)} \Gamma(x-y)\varphi_{,i}(x)\ dx \\ &= -\int_{\mathbb{R}^n\setminus \bar{B}_\epsilon(y)} \Gamma_{,i}(x-y)\varphi(x)\ dx \\ &\quad + \int_{\partial B_\epsilon(y)} \Gamma(x-y)\varphi(x)\left\langle \frac{y-x}{|y-x|}, e_i\right\rangle\ dx \\ &\quad + \int_{\bar{B}_\epsilon(y)} \Gamma(x-y)\varphi_{,i}(x)\ dx \\ &\to -\int_{\mathbb{R}^n} \Gamma_{,i}(x-y)\varphi(x)\ dx\end{aligned}$$

as $\epsilon \to 0$, since all functions are integrable. The weak differentiability follows. To prove the second claim we show

$$\int_{\mathbb{R}^n} \Gamma(x-y)\Delta\varphi(x)\ dx = \varphi(y) \quad \forall \varphi \in C_c^2(\mathbb{R}^n).$$

Since $\Gamma(\cdot - y)$ has a singularity at y, we can not just apply partial integration. First we must *cut out the singularity* and then perform a detailed analysis at the point y. With the help of the second Green's formula, Exercise 1.4.6, we deduce

$$\begin{aligned}&\int_{\mathbb{R}^n} \Gamma(x-y)\Delta\varphi(x)\ dx \\ =\ &\int_{\mathbb{R}^n\setminus \bar{B}_\epsilon(y)} \Gamma(x-y)\Delta\varphi(x)\ dx + \int_{\bar{B}_\epsilon(y)} \Gamma(x-y)\Delta\varphi(x)\ dx \\ =\ &-\int_{\mathbb{R}^n\setminus \bar{B}_\epsilon(y)} \langle \nabla_x\Gamma(x-y), \nabla\varphi(x)\rangle\ dx + \int_{\partial B_\epsilon(y)} \Gamma(\cdot - y)\langle \nabla\varphi, \nu\rangle \\ &+ \int_{\bar{B}_\epsilon(y)} \Gamma(x-y)\Delta\varphi(x)\ dx.\end{aligned} \tag{5.1}$$

The second and third term on the right hand side will vanish in the limit as $\epsilon \to 0$. Hence we only have to investigate the first term:

$$\begin{aligned}-\int_{\mathbb{R}^n\setminus \bar{B}_\epsilon(y)} \langle \nabla_x\Gamma(\cdot - y), \nabla\varphi\rangle &= -\int_{\partial B_\epsilon(y)} \varphi(x)\left\langle \nabla_x\Gamma(x-y), \frac{y-x}{|y-x|}\right\rangle\ dx \\ &= \frac{1}{n\omega_n}\int_{\partial B_\epsilon(y)} \varphi(x)\frac{1}{|y-x|^{n-1}}\ dx \\ &\to \varphi(y).\end{aligned}$$

Letting $\epsilon \to 0$ in (5.1) gives the result. □

5.1.8 Corollary (Representation formula for Δ). *Let $2 \le n \in \mathbb{N}$ and $f \in C_c^2(\mathbb{R}^n)$. Then*

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y)f(y)\ dy$$

defines a $C^2(\mathbb{R}^n)$-function and solves

$$\Delta u = f.$$

Moreover, if $n \geq 3$, every bounded solution $u \in C^2(\mathbb{R}^n)$ of $\Delta u = f$ has this form up to a constant.

Proof.

$$u(x) = \int_{\mathbb{R}^n} \Gamma(z) f(x-z)\ dz.$$

Hence $u \in C^2(\mathbb{R}^n)$ and the result follows from Proposition 5.1.7. The uniqueness follows from the Liouville theorem. Note that in case $n = 2$, u does not need to be bounded. ☐

Green's function

Corollary 5.1.8 provides a way to explicitly write down a solution to

$$\Delta u = f,$$

on the whole $\mathbb{R}^n$, once the function $f \in C_c^2(\mathbb{R}^n)$ is known. But what about the Dirichlet problem in domains Ω with given boundary values? We want to construct a similar integration kernel as Γ, such that we get a representation formula for solutions of the Dirichlet problem on a domain, i.e. we want to give a formula for the solution to

$$\begin{aligned} \Delta u &= f \\ u_{|\partial\Omega} &= g, \end{aligned} \tag{5.2}$$

where $\partial\Omega \in C^\infty$ and $g \in C^\infty(\bar{\Omega})$. The obvious strategy is to use Γ to get a solution to $\Delta u = f$ and then use a *correction term* to adjust the boundary values. This correction term shall not destroy $\Delta u = f$ again, so we want to make it harmonic.

5.1.9 Definition (Green's function for a domain)**.** Let $2 \leq n \in \mathbb{N}$ and $\Omega \Subset \mathbb{R}^n$ with smooth boundary. The *Green's function for* Ω is defined by

$$G(x,y) = \Gamma(x-y) - \phi^x(y), \quad x, y \in \Omega, x \neq y,$$

where ϕ^x is the unique solution of

$$\begin{aligned} \Delta \phi^x &= 0 \quad \text{in } \Omega \\ \phi^x &= \Gamma(\cdot - x) \quad \text{on } \partial\Omega. \end{aligned}$$

We obtain

5.1.10 Proposition. *Let $2 \leq n \in \mathbb{N}$ and $\Omega \Subset \mathbb{R}^n$ with smooth boundary. Let $f, g \in C^\infty(\bar{\Omega})$. Then the function*

$$u(x) = \int_\Omega G(x,y) f(y)\ dy + \int_{\partial\Omega} g(y) \frac{\partial G}{\partial \nu_y}(x,y)\ dy$$

solves eq. (5.2).

Proof. We already know a solution to exist, so call it u. Let $x \in \Omega$. From the second Green's identity applied to the domain $\Omega \backslash \bar{B}_\epsilon(x)$ we obtain

$$\begin{aligned}
\int_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu_y}(x,y)\;dy &= \int_{\Omega\backslash \bar{B}_\epsilon(x)} \langle \nabla u, \nabla G(x,\cdot)\rangle - \int_{\partial B_\epsilon(x)} u(y)\frac{\partial G}{\partial \nu_y}(x,y)\;dy \\
&= -\int_{\Omega\backslash \bar{B}_\epsilon(x)} f(y)G(x,y)\;dy + \int_{\partial B_\epsilon(x)} \frac{\partial u}{\partial \nu}G(x,\cdot) \\
&\quad - \int_{\partial B_\epsilon(x)} u(y)\frac{\partial G}{\partial \nu_y}(x,y)\;dy \\
&\to -\int_{\Omega} f(y)G(x,y)\;dy + u(x)
\end{aligned}$$

as $\epsilon \to 0$. Hence u is given by the desired formula. □

5.1.11 Remark. (i) Of course this representation formula is not very explicit, since it involves the construction of a harmonic function with given boundary values. However, it is completely determined by the fundamental solution. For special domains such as a half space and balls one can write down the G explicitly.

(ii) The regularity assumptions of the data can usually be weakened. However, since we apply the regularity and existence theory in order to get existence of a Green's function, we stick to the smooth case in the above definitions.

5.1.12 Exercise (Green's function for a ball)**.** Let $2 \le n \in \mathbb{N}$, $0 \neq x \in \mathbb{R}^n$ and denote by

$$\bar{x} = \frac{R^2}{|x|^2}x$$

the *inversion* at $\partial B_R(0)$.

(i) Prove that

$$G(x,y) = \begin{cases} \Gamma(|x-y|) - \Gamma\left(\frac{|y|}{R}|x-\bar{y}|\right), & y \neq 0 \\ \Gamma(|x|) - \Gamma(R), & y = 0 \end{cases}$$

is the Green's function for $B_R(0)$.

(ii) Let $g \in C^0(\partial\Omega)$. Prove that

$$u(x) = \begin{cases} \frac{R^2-|x|^2}{n\omega_n R}\int_{\partial B_R(0)} \frac{g(y)}{|x-y|^n}\;dy, & x \in B_R(0) \\ g(x), & x \in \partial B_R(0) \end{cases}$$

defines a harmonic function with boundary values g.

Remark: Note that by putting $x = 0$ we recover the mean value property of harmonic functions.

Perron's method

Perron's method [13] provides a very elegant and powerful way to construct harmonic functions on a domain with given continuous boundary values. It does not rely on any previous results except the maximum principle for harmonic functions and the solvability of the Dirichlet problem in balls. The presentation follows [5]. We need several preliminary definitions and results.

5.1.13 Definition (Sub- and Superharmonic functions)**.** Let $2 \leq n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ open. A function $u \in C^0(\Omega)$ is called *subharmonic*, if for every ball $\bar{B} \subset \Omega$ and every harmonic function $h \in C^\infty(\bar{B})$ there holds

$$u_{|\partial B} \leq h_{|\partial B} \quad \Rightarrow \quad u \leq h.$$

u is called *superharmonic*, if $-u$ is subharmonic.

5.1.14 Exercise (Classical subharmonic functions)**.** Let $2 \leq n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ open. Suppose $u \in C^2(\Omega)$ is subharmonic. Prove that

$$\Delta u \geq 0.$$

5.1.15 Proposition. *Let $2 \leq n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be a domain. Let $v \in C^0(\bar{\Omega})$ superharmonic and $u \in C^0(\bar{\Omega})$ subharmonic. Then there hold*

(i) u satisfies the strong maximum principle in Ω.

(ii)

$$v_{|\partial\Omega} \geq u_{|\partial\Omega} \quad \Rightarrow \quad \left(v_{|\Omega} > u_{|\Omega} \quad or \quad v \equiv u\right).$$

Proof. It is enough to prove (ii). The set

$$A = \{x \in \Omega \colon u(x) - v(x) = \sup_{\Omega}(u - v)\}.$$

is closed in Ω. If it was not open, then there existed $x \in A$ and a ball $B \subset \Omega$ around x with

$$(u - v)_{|\partial B} \not\equiv M.$$

Denote by $\bar{u}$ and $\bar{v}$ the harmonic extension of u and v in B. Then

$$M \geq \max_{\partial B}(\bar{u} - \bar{v}) \geq \bar{u}(x) - \bar{v}(x) \geq u(x) - v(x) = M.$$

Hence $\bar{u} - \bar{v} \equiv M$, a contradiction. Hence, if the function $u - v$ attains an interior maximum, it must be constant. In this case the constant must be negative or zero due to the boundary condition. In the other case we must have $v > u$. □

5.1.16 Definition (Harmonic lifting)**.** Let $2 \leq n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open, $u \in C^0(\Omega)$ subharmonic and $B \Subset \Omega$ a ball. Define the *harmonic lifting of u in B* by

$$U(x) = \begin{cases} \bar{u}(x), & x \in B \\ u(x), & x \in \Omega \backslash B, \end{cases}$$

where $\bar{u}$ is the harmonic extension of $u_{|\partial B}$.

5.1.17 Proposition. *Let $2 \leq n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open, $u_1, \ldots, u_N \in C^0(\Omega)$ subharmonic and $B \Subset \Omega$ a ball. Then there hold:*

(i) The function $v = \max(u_1, \dots, u_N) \in C^0(\Omega)$ is subharmonic.

(ii) The harmonic lifting U of u in B is subharmonic.

Proof. (i): Let $B' \Subset \Omega$ a ball and a harmonic function h with

$$v_{|\partial B'} \leq h_{|\partial B'}.$$

Then this carries over to the u_i and hence

$$v \leq h.$$

(ii): Let $B' \Subset \Omega$ a ball and a harmonic function h with

$$U_{|\partial B'} \leq h_{|\partial B'}.$$

Since $u \leq U$, we obtain

$$u_{|B'} \leq h.$$

There hold

$$\partial(B \cap B') = (\partial B \cap \bar{B}') \cup (\partial B' \cap \bar{B})$$

and

$$\bar{u}_{|\partial B' \cap \bar{B}} \leq h, \quad \bar{u}_{|\partial B \cap \bar{B}'} = u_{|\partial B \cap \bar{B}'} \leq h.$$

The maximum principle implies

$$\bar{u}_{|B'} \leq h.$$

Hence $U_{B'} \leq h$. □

5.1.18 Definition (Subfunctions). Let $2 \leq n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ open and $\varphi \in \mathbb{R}^{\partial\Omega}$ bounded.

(i) A subharmonic function $u \in C^0(\bar{\Omega})$ is called *subfunction rel* φ, if $u_{|\partial\Omega} \leq \varphi$. Denote by S_φ the set of all subfunctions rel φ.

(ii) A superharmonic function $u \in C^0(\bar{\Omega})$ is called *superfunction rel* φ, if $u_{|\partial\Omega} \geq \varphi$.

The following theorem constructs a harmonic function on Ω.

5.1.19 Theorem (Perron). *Let $2 \leq n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ open and $\varphi \in \mathbb{R}^{\partial\Omega}$ bounded. Then the function*

$$u(x) = \sup_{v \in S_\varphi} v(x)$$

is harmonic.

Proof. All $v \in S_\varphi$ are bounded, because

$$v_{|\partial\Omega} \leq \sup_{\partial\Omega} \varphi$$

and the latter function is harmonic. Hence u is real-valued function. Let $x \in \Omega$, then there exists a sequence of subharmonic functions $(\tilde{v}_n)_{n \in \mathbb{N}}$ in S_φ with

$$\tilde{v}_n(x) \to u(x).$$

Set

$$v_n = \max(\tilde{v}_n, \inf \varphi).$$

Then also $v_n(x) \to u(x)$. Fix a ball $x \in B \Subset \Omega$ and let $V_n \in S_\varphi$ be the harmonic lifting of v_n in B. Then

$$V_n(x) \to u(x)$$

and V_n is uniformly bounded. Due to Exercise 5.1.4 there exists a subsequence $(V_n)_{n\in\mathbb{N}}$ such that for all $B' \Subset B$

$$|V_n - v|_{0,B'} \to 0,$$

where v is harmonic in B. There hold

$$v \le u, \quad v(x) = u(x).$$

Claim: $v = u$ in B. Otherwise there existed a point $y \in B$ with $v(y) < u(y)$ and a function $\tilde{u} \in S_\varphi$ with

$$v(y) < \tilde{u}(y) \le u(y). \tag{5.3}$$

Let $w_n = \max(\tilde{u}, V_n)$ and W_n the corresponding harmonic liftings in B. Again, a subsequence $(W_n)_{n\in\mathbb{N}}$ converges uniformly in any $B' \Subset B$ to a harmonic function w with

$$v \le w \le u, \quad v(x) = w(x) = u(x).$$

The maximum principle implies $v = w$ in B, in contradiction to (5.3). Hence u is harmonic. □

To investigate, under which assumptions on the boundary $\partial\Omega$ the function φ is attained continuously by the harmonic function u, we make the following definition.

5.1.20 Definition (Barriers). Let $2 \le n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ and $\xi \in \partial\Omega$. A function $w \in C^0(\bar{\Omega})$ is called a *barrier at ξ relative to Ω*, if

(i) w is superharmonic

(ii) $w(\xi) = 0$ and $w > 0$ on $\bar{\Omega}\backslash\{\xi\}$.

ξ is called *regular*, if there exists a barrier at ξ relative to Ω.

5.1.21 Lemma. *Let $2 \le n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$, $\xi \in \partial\Omega$ regular and $\varphi \in \mathbb{R}^{\partial\Omega}$ bounded, such that φ is continuous at ξ. Let u be the function defined in Theorem 5.1.19. Then*

$$\lim_{x\to\xi} u(x) = \varphi(\xi).$$

Proof. Let $\epsilon > 0$. Then there exists a constant $k > 0$, such that $\varphi(\xi) + \epsilon + kw$ is a superfunction rel φ and such that $\varphi(\xi) - \epsilon - kw$ is a subfunction rel φ. Since u is harmonic, we deduce

$$\varphi(\xi) - \epsilon - kw \le u \le \varphi(\xi) + \epsilon + kw$$

and hence

$$|u(x) - \varphi(\xi)| \le \epsilon + kw(x).$$

The result follows, since $w(x) \to 0$ as $x \to \xi$. □

The following theorem is the main conclusion of Perron's method.

5.1.22 Theorem. *Let $2 \le n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$. Then the classical Dirichlet problem*

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega \\ u &= \varphi \quad \text{on } \partial\Omega \end{aligned}$$

is solvable in $C^\infty(\Omega) \cap C^0(\bar{\Omega})$ for arbitrary $\varphi \in C^0(\partial\Omega)$ if and only if every boundary point of $\partial\Omega$ is regular.

Proof. Given that every boundary point is regular, the solvability follows from Lemma 5.1.21. Conversely, suppose that the problem is solvable for any given φ. Let $\xi \in \partial\Omega$ and put

$$\varphi(x) = |x - \xi|.$$

The solution corresponding to these boundary values is a barrier at ξ. □

Theoretically this is a nice result, but in practice it will only be useful if we have a simple criterion, when a boundary point is regular. Otherwise we will not be able to decide when the Dirichlet problem is solvable. Fortunately there is such a criterion:

5.1.23 Proposition. *Let $2 \le n \in \mathbb{N}$ and $\Omega \Subset \mathbb{R}^n$ satisfy an exterior ball condition at every $\xi \in \partial\Omega$.*[2] *Then every boundary point is regular.*

Proof. Set

$$w(x) = \begin{cases} \log\left(\frac{|x-y|}{R}\right), & n = 2 \\ R^{2-n} - |x-y|^{2-n}, & n \ge 3, \end{cases}$$

where $B_R(y)$ is an exterior ball at ξ. Then $w(\xi) = 0$ and $w(x) > 0$ for all $x \in \bar{\Omega} \backslash \{\xi\}$. Furthermore w is harmonic, so that w is a barrier. □

5.2 Heat equation

In this section we will obtain first classical existence results for the heat equation

$$\Delta_x u - \dot{u} = 0. \tag{5.4}$$

This is the prototype of a linear parabolic PDE. Due to the maximum principle for parabolic equations, Theorem 2.2.2, it seems natural to consider (5.4) on a domain $Q \subset \mathbb{R}^{n+1}$ with prescribed parabolic boundary values, namely we consider the so-called *Cauchy-Dirichlet problem*

$$\begin{aligned} \Delta_x u - \dot{u} &= 0 \quad \text{in } Q \\ u &= \varphi \quad \text{on } \partial_P Q, \end{aligned}$$

where $Q = (0, T) \times \Omega$ is a cylinder with open $\Omega \subset \mathbb{R}^n$. We shall first consider $Q = \mathbb{R}^{n+1}_+ \equiv \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t > 0\}$. In this special case we can define a fundamental solution, similarly to the Laplace equation. The major source for the first part of this section is [1].

[2] This means that $\mathbb{R}^n \backslash \bar{\Omega}$ satisfies an interior ball condition, compare Definition 2.2.13.

The fundamental solution

5.2.1 Definition. Let $n \in \mathbb{N}$. The function

$$\Gamma\colon (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$$
$$\Gamma(t,x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x|^2}{4t}\right)$$

is called the *fundamental solution of the heat equation* or also the *heat kernel.*

The following observation justifies the terminology.

5.2.2 Theorem. *Let $n \in \mathbb{N}$. The heat kernel Γ satisfies*

$$\int_{\mathbb{R}^n} \Gamma(t,x)\ dx = 1 \quad \forall t > 0$$

and

$$\Delta_x \Gamma - \dot{\Gamma} = 0.$$

Proof. There holds, using the transformation of variables $y = \frac{x}{2\sqrt{t}}$,

$$\int_{\mathbb{R}^n} \Gamma(t,x)\ dx = \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-|y|^2}\ dy = \frac{1}{\pi^{\frac{n}{2}}} \left(\int_{-\infty}^{\infty} e^{-x^2}\ dx\right)^n = 1.$$

Moreover,

$$\dot{\Gamma} = -\frac{n}{2t}\Gamma + \frac{|x|^2}{4t^2}\Gamma = \Delta_x \Gamma.$$

□

Now we are able to solve the Cauchy problem for the half space.

5.2.3 Theorem. *Let $\varphi \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then a solution to*

$$\begin{aligned} \Delta_x u - \dot{u} &= 0 \quad \text{in } \mathbb{R}^{n+1}_+ \\ u &= \varphi \quad \text{on } \{0\} \times \mathbb{R}^n \end{aligned}$$

in $C^\infty(\mathbb{R}^{n+1}_+) \cap C^0(\bar{\mathbb{R}}^{n+1}_+)$ is given by

$$u(t,x) = \int_{\mathbb{R}^n} \Gamma(t, x-\xi)\varphi(\xi)\ d\xi. \tag{5.5}$$

Proof. Since the integral converges locally in $\mathbb{R}^{n+1}_+$ uniformly and Γ is smooth, differentiation under the integral is justified. Thus u is a solution of the heat equation. We have to check the continuity at $t = 0$. We calculate for every

$t > 0$, $\delta > 0$ and $|x - x_0| < \frac{\delta}{2}$:

$$\begin{aligned}
|u(x,t) - \varphi(x_0)| &\leq \int_{\mathbb{R}^n} \Gamma(t, x-\xi)|\varphi(\xi) - \varphi(x_0)|\; d\xi \\
&= \int_{B_\delta(x_0)} \Gamma(t, x-\xi)|\varphi(\xi) - \varphi(x_0)|\; d\xi \\
&\quad + \int_{\mathbb{R}^n \setminus B_\delta(x_0)} \Gamma(t, x-\xi)|\varphi(\xi) - \varphi(x_0)|\; d\xi \\
&\leq \operatorname{osc}_{B_\delta(x_0)}(\varphi) + 2|\varphi|_{0,\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B_\delta(x_0)} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|\xi - x_0|^2}{16t}}\; d\xi \\
&\leq \operatorname{osc}_{B_\delta(x_0)}(\varphi) + c_n|\varphi|_{0,\mathbb{R}^n} \int_{\frac{\delta}{4\sqrt{t}}}^\infty e^{-r^2} r^{n-1}\; dr < \epsilon,
\end{aligned}$$

whenever $\delta = \delta(\epsilon)$ and $t = t(\delta)$ are small. □

Note that the solution (5.5) is positive *everywhere* instantly, if the initial datum is positive *somewhere*. This phenomenon is known as infinite speed of propagation.

Now we want to solve the inhomogeneous problem, i.e.

$$\begin{aligned}
\Delta_x u - \dot{u} &= f \quad \text{in } \mathbb{R}^{n+1}_+ \\
u &= 0 \quad \text{on } \{0\} \times \mathbb{R}^n.
\end{aligned} \tag{5.6}$$

Motivated by Theorem 5.2.3 and the fundamental theorem of calculus, we expect the following proposition to hold.

5.2.4 Proposition (Duhamel's principle). *Let $n \in \mathbb{N}$ and $f \in C^{1;2}_c([0,\infty) \times \mathbb{R}^n)$. Then the function u defined by*

$$u(t,x) = -\int_0^t \int_{\mathbb{R}^n} \Gamma(t-s, x-\xi) f(s,\xi)\; ds d\xi$$

solves (5.6) and there holds $u \in C^\infty((0,\infty) \times \mathbb{R}^n) \cap C^0(\bar{\mathbb{R}}^{n+1}_+)$, with $u(0,x) = 0$ for all $x \in \mathbb{R}^n$.

Proof. A change of variables gives

$$u(t,x) = -\int_0^t \int_{\mathbb{R}^n} \Gamma(\tau, y) f(t-\tau, x-y)\; d\tau dy.$$

Thus

$$\begin{aligned}
&\Delta_x u - \dot{u} \\
&= \int_0^t \int_{\mathbb{R}^n} \Gamma(\tau,y)(\partial_t - \Delta_x) f(t-\tau, x-y)\; d\tau dy + \int_{\mathbb{R}^n} \Gamma(t,y) f(0, x-y)\; dy \\
&= \int_\epsilon^t \int_{\mathbb{R}^n} \Gamma(\tau,y)(-\Delta_y - \partial_\tau) f(t-\tau, x-y)\; d\tau dy \\
&\quad + \int_0^\epsilon \int_{\mathbb{R}^n} \Gamma(\tau,y)(\partial_t - \Delta_x) f(t-\tau, x-y)\; d\tau dy + \int_{\mathbb{R}^n} \Gamma(t,y) f(0, x-y)\; dy \\
&\leq \int_{\mathbb{R}^n} \Gamma(\epsilon, y) f(t-\epsilon, x-y)\; dy + \epsilon C \\
&\to f(t,x)
\end{aligned}$$

for $\epsilon \to 0$, as in the proof of Theorem 5.2.3. $\square$

Combining these results gives:

5.2.5 Theorem. *Let $n \in \mathbb{N}$, $f \in C_c^{1;2}([0,\infty)\times\mathbb{R}^n)$ and $\varphi \in C^0(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. Then the function u defined by*

$$u(t,x) = \int_{\mathbb{R}^n} \Gamma(t, x-\xi)\varphi(\xi)\, d\xi - \int_0^t \int_{\mathbb{R}^n} \Gamma(t-s, x-\xi) f(s,\xi)\, dsd\xi$$

belongs to $u \in C^\infty((0,\infty)\times\mathbb{R}^n)\cap C^0(\bar{\mathbb{R}}^{n+1}_+)$ and solves

$$\begin{aligned} \Delta_x u - \dot{u} &= f \quad \text{in } \mathbb{R}^{n+1}_+ \\ u &= \varphi \quad \text{on } \{0\}\times\mathbb{R}^n. \end{aligned}$$

The Cauchy-Dirichlet problem in domains

The following method to construct solutions to the heat equation in arbitrary domains only relies on the *elliptic* L^2-theory and hence is quite elegant. As a motivation note the following fact. If $\Omega \Subset \mathbb{R}^n$ has a smooth boundary and $v \in C^\infty(\Omega)\cap W_0^{1,2}(\Omega)$ is one of the Laplace eigenfunctions,

$$-\Delta v = \lambda v,$$

then the function

$$u(t,x) = e^{-\lambda t} v(x)$$

is a solution to the heat equation in $(0,T)\times\Omega$ with zero Dirichlet boundary conditions. Hence, since any $L^2(\Omega)$ function u_0 can be expanded as a Fourier series,

$$u_0 = \sum_{i=1}^\infty \langle u_0, u_i\rangle_{2,\Omega}\, u_i,$$

where $(u_i)_{i\in\mathbb{N}}$ is the countable family of normalized Laplace-eigenfunctions corresponding to the eigenvalues λ_i, we can expect using formal[3] differentiation, that

$$u(t,x) = \sum_{i=1}^\infty e^{-\lambda_i t}\langle u_0, u_i\rangle_{2,\Omega}\, u_i$$

will solve the Cauchy-Dirichlet problem

$$\begin{aligned} \Delta_x - \dot{u} &= 0 \quad \text{in } (0,\infty)\times\Omega \\ u(t,\cdot) &= 0 \quad \text{on } \partial\Omega \quad \forall t>0 \\ u(0,\cdot) &= u_0. \end{aligned}$$

Note that from these equations we would then obtain

$$\Delta_x^m u(t,x) = \frac{d^m}{dt^m} u(t,x) = 0 \quad \forall x \in \partial\Omega\ \forall t>0\ \forall m \geq 0.$$

[3] yet unjustified

Hence, to even have a chance to make the solution u belong to the class $C^\infty([0,T)\times\bar{\Omega})$, we have to impose the *compatibility condition*

$$\Delta^m u_{0|\partial\Omega} = 0 \quad \forall m \geq 0.$$

In this section we make this plausible argument rigorous. We start with weak initial values.

5.2.6 Theorem. *Let $n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ with smooth boundary and $u_0 \in L^2(\Omega)$. Let u_i be an $L^2(\Omega)$-orthonormal basis in $W^{1,2}_0(\Omega)$ of Laplace eigenfunctions with eigenvalues λ_i. Then the function*

$$u\colon (0,\infty)\times\Omega \to \mathbb{R}$$
$$u(t,x) = \sum_{i=1}^{\infty} e^{-\lambda_i t}\,\langle u_0, u_i\rangle_{2,\Omega}\, u_i(x)$$

is a smooth solution to the heat equation. Furthermore there holds

$$\|u(t,\cdot) - u_0\|_{2,\Omega} \to 0, \quad t \to 0$$

and if in addition $\Delta^{m-1}u_0 \in W^{2,2}(\Omega)\cap W^{1,2}_0(\Omega)$ for $m \geq 1$, then there also holds

$$\|u(t,\cdot) - u_0\|_{2m,2,\Omega} \to 0, \quad t \to 0.$$

Proof. Recall that $u_i \in C^\infty(\bar{\Omega})$. Since the satisfy

$$-\Delta u_i = \lambda_i u_i,$$

the L^2-regularity theory, in particular Theorem 4.2.10, gives

$$\|u_i\|_{m+2,2,\Omega} \leq c(\|u_i\|_{2,\Omega} + \lambda_i\|u_i\|_{m,2,\Omega})$$

for all large m and an induction and the Sobolev embedding theorems give

$$|u_i|_{k,\Omega} \leq c\|u_i\|_{m+2,2,\Omega} \leq c(1+\lambda_i^{m+1}).$$

We prove that the partial sums

$$v_N = \sum_{i=1}^{N} e^{-\lambda_i t}\,\langle u_0, u_i\rangle_{2,\Omega}\, u_i$$

form a Cauchy sequence in $C^k(\bar{\Omega})$ for any $k \geq 1$ and give $t > 0$. Therefore pick $m = m(k)$ large enough to ensure

$$W^{m,2}(\Omega) \hookrightarrow C^k(\bar{\Omega}).$$

There holds

$$
\begin{aligned}
&\left\| \sum_{i=N}^{M} e^{-\lambda_i t} \langle u_0, u_i \rangle_{2,\Omega} u_i \right\|_{m,2,\Omega}^2 \\
= &\sum_{i=N}^{M} e^{-2\lambda_i t} | \langle u_0, u_i \rangle_{2,\Omega} |^2 \|u_i\|_{m,2,\Omega}^2 \\
&+ \sum_{N \le i,j \le M} e^{-(\lambda_i + \lambda_j)t} | \langle u_0, u_i \rangle_{2,\Omega} || \langle u_0, u_j \rangle_{2,\Omega} | \langle u_i, u_j \rangle_{m,2,\Omega} \\
&\le \sum_{i=N}^{M} p(\lambda_i) e^{-\lambda_i t} | \langle u_0, u_i \rangle_{2,\Omega} |^2 \\
&\to 0,
\end{aligned}
$$

as $N, M \to \infty$. Here p is a polynomial. The convergence is uniform on each $[\epsilon, \infty) \subset (0, \infty)$.[4] Hence

$$
v = \sum_{i=1}^{\infty} e^{-\lambda_i t} \langle u_0, u_i \rangle_{2,\Omega} u_i \in C^\infty(\bar{\Omega}),
$$

the differentiation under the summation sign is justified and the first claim of the proposition follows. For the second claim we estimate

$$
\begin{aligned}
\|u(t, \cdot) - u_0\|_{2,\Omega} \le &\left\| \sum_{i=1}^{N} (1 - e^{-\lambda_i t}) \langle u_0, u_i \rangle_{2,\Omega} u_i \right\|_{2,\Omega} \\
&+ \left\| \sum_{i=N+1}^{\infty} (1 - e^{-\lambda_i t}) \langle u_0, u_i \rangle_{2,\Omega} u_i \right\|_{2,\Omega}.
\end{aligned}
$$

First pick N large enough to ensure that the second term is less than a given $\epsilon > 0$. Then let $t \to 0$.

Now let $\Delta^{m-1} u_0 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ for all $m \ge 1$. First note that the L^2-estimates imply that for all functions $v \in W^{m+2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ there holds

$$
\|v\|_{m+2,2,\Omega} \le c(\|v\|_{2,\Omega} + \|\Delta v\|_{m,2,\Omega}).
$$

Thus for $t > 0$,

$$
\|u(t, \cdot) - u_0\|_{2m,2,\Omega} \le c(\|u(t, \cdot) - u_0\|_{2,\Omega} + \|\Delta(u(t, \cdot) - u_0)\|_{2m-2,2,\Omega})
$$

[4] We have used Parseval's inequality for orthonormal basis of Hilbert spaces, i.e. if

$$
x = \sum_{i=1}^{\infty} \langle x, u_i \rangle u_i,
$$

then

$$
\sum_{i=1}^{\infty} | \langle x, u_i \rangle |^2 < \infty.
$$

and hence by induction

$$\|u(t,\cdot)-u_0\|_{2m,2,\Omega} \le c\sum_{k=0}^{m} \|\Delta^k(u(t,\cdot)-u_0)\|_{2,\Omega}.$$

We estimate each term on the right hand side:

$$\begin{aligned}
\|\Delta^k(u(t,\cdot)-u_0)\|_{2,\Omega} &= \left\| \sum_{i=1}^{\infty} \lambda_i^k (1-e^{-\lambda_i t}) \langle u_0, u_i\rangle_{2,\Omega} u_i \right\|_{2,\Omega} \\
&\le \left\| \sum_{i=1}^{N} \lambda_i^k (1-e^{-\lambda_i t}) \langle u_0, u_i\rangle_{2,\Omega} u_i \right\|_{2,\Omega} \\
&\quad + \left\| \sum_{i=N+1}^{\infty} \lambda_i^k (1-e^{-\lambda_i t}) \langle u_0, u_i\rangle_{2,\Omega} u_i \right\|_{2,\Omega},
\end{aligned}$$

which, by the same reasoning as above, converges to zero as $t \to 0$. That the latter term becomes small, when N approaches infinity, is due to the fact that the Fourier series of $-\Delta^k u_0$ is given by

$$-\Delta^k u_0 = \sum_{i=1}^{\infty} \left\langle -\Delta^k u_0, u_i \right\rangle_{2,\Omega} u_i = \sum_{i=1}^{\infty} \lambda_i^k \langle u_0, u_i\rangle u_i,$$

which shows that the latter series converges in $L^2(\Omega)$. The proof is complete. □

5.2.7 Corollary. *Let $n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ with smooth boundary, $u_0 \in C^\infty(\bar{\Omega})$, such that $\Delta^m u_0 \in W_0^{1,2}(\Omega)$ for all $m \in \mathbb{N}$. Let u_i be an $L^2(\Omega)$-orthonormal basis in $W_0^{1,2}(\Omega)$ of Laplace eigenfunctions with eigenvalues λ_i. Then the function*

$$u\colon Q = (0,\infty)\times\Omega \to \mathbb{R}$$

$$u(t,x) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \langle u_0, u_i\rangle_{2,\Omega} u_i(x)$$

is the unique $C^\infty(\bar{Q})$-solution to the Cauchy-Dirichlet problem

$$\begin{aligned}
\Delta_x u - \dot{u} &= 0 \quad \text{in } Q \\
u(t,\cdot) &= 0 \quad \text{on } \partial\Omega \quad \forall t \ge 0 \\
u(0,\cdot) &= u_0.
\end{aligned}$$

Proof. Due to the uniform convergence of the integral in $\{t \ge \epsilon\}$ there holds $u \in C^\infty((0,\infty)\times\bar{\Omega})$. From the Sobolev embedding and Theorem 5.2.6 we obtain

$$|u(t,\cdot)-u_0|_{k,\bar{\Omega}} \to 0, \quad t \to 0.$$

Hence all spatial derivatives of $u(t,\cdot)$ can be continuously extended to $\bar{Q}$. We have to prove the same for the time derivative. There holds for all $k > 0$

$$\frac{d^k}{dt^k} u(t,x) = \Delta_x^k u(t,x) \to \Delta^k u_0(x), \quad t \to 0$$

and hence $u \in C^\infty(\bar{Q})$. □

Harnack inequality

The Harnack inequality for the heat equation is a little bit different that for harmonic functions. As expected, a heat distribution will need a little bit of time to level out along space. Hence in order to estimate the maximal heat by the minimal heat, we have to wait for a little while. Thus all we should expect is an estimate of the rough form

$$\sup_{M_{t_1}} u \leq c(t_1, t_2) \inf_{M_{t_2}} u,$$

where M_{t_i} are some compact subsets of Ω at times $t_1 < t_2$. For the heat equation such an estimate was first proven independently by Hadamard and Pini, [6, 14]. We present an elegant proof due to Li and Yau, [10], which is applicable to many other parabolic operators, even nonlinear ones.

5.2.8 Theorem (Harnack inequality for the heat equation). *Let $n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ open, $T > 0$ and $u \in C^\infty((0,T) \times \Omega)$ a positive solution of the heat equation. Then for every ball $B_r(x_0) \Subset \Omega$ and times $0 < t_1 < t_2 < T$ there exists a constant c, such that*

$$\max_{\bar{B}_r(x_0)} u(t_1, \cdot) \leq c \min_{\bar{B}_r(x_0)} u(t_2, \cdot).$$

Proof. We will prove that the function

$$v = \log u$$

satisfies the inequality

$$\dot{v} \geq \alpha |\nabla v|^2 - \beta \tag{5.7}$$

in $[t_1, t_2] \times \bar{B}_r(x_0)$ with suitable constant α and β. From this we will conclude the Harnack inequality as follows.

$$\begin{aligned} v(t_2, x_2) - v(t_1, x_1) &= \int_0^1 \frac{d}{ds} v(st_2 + (1-s)t_1, sx_2 + (1-s)x_1)\ ds \\ &= \int_0^1 (\dot{v}(t_2 - t_1) + \langle \nabla v, x_2 - x_1 \rangle)\ ds \\ &\geq \int_0^1 \alpha |\nabla v|^2 (t_2 - t_1) - \beta(t_2 - t_1) + \langle \nabla v, x_2 - x_1 \rangle\ ds \\ &\geq -c, \end{aligned}$$

where c only depends on r, $t_2 - t_1$, α and β. Hence

$$u(t_2, x_2) \geq e^{-c} u(t_1, x_1).$$

Taking the maximum over x_1 and the minimum over x_2 gives the Harnack inequality.

Hence we have to prove (5.7). v satisfies

$$\Delta_x v - \dot{v} = -|\nabla v|^2.$$

For $\kappa > 0$ to be determined later we obtain an equation for

$$q = \Delta_x v + \kappa |\nabla v|^2,$$

namely

$$
\begin{aligned}
\Delta_x q - \dot{q} &= \Delta_x(\Delta_x v) + 2\kappa \langle \nabla(\Delta_x v), \nabla v\rangle + 2\kappa|\nabla^2 v|^2 - \Delta_x \dot{v} - 2\kappa \langle \nabla\dot{v}, \nabla v\rangle \\
&= 2(\kappa-1)\langle \nabla(\Delta_x v), \nabla v\rangle + 2(\kappa-1)|\nabla^2 v|^2 - 2\kappa \langle \nabla(\Delta_x v), \nabla v\rangle \\
&\quad - 2\kappa \left\langle \nabla|\nabla v|^2, \nabla v\right\rangle \\
&= -2\langle \nabla q, \nabla v\rangle + 2(\kappa-1)|\nabla^2 v|^2.
\end{aligned}
$$

Now let ζ be a smooth cut-off function with

$$
\zeta_{|\partial_p Q} = 0, \quad \zeta_{|(0,T)\times\Omega} > 0, \quad \zeta_{|[t_1,t_2]\times \bar{B}_r(x_0)} = 1.
$$

Set, for $\mu > 0$,

$$
z = \zeta^4 q + \mu t
$$

and suppose z attains a negative minimum at some interior point, at which we firstly conclude

$$
|\nabla v|^2 \le \frac{1}{\kappa}|\Delta_x v| \le \frac{c}{\kappa}|\nabla^2 v|
$$

and thus

$$
|q| \le \frac{c}{\kappa}|\nabla^2 v|,
$$

secondly

$$
0 = \frac{\nabla z}{\zeta^3} = 4\nabla\zeta q + \zeta\nabla q
$$

and thirdly, putting $\kappa = \frac{1}{2}$,

$$
\begin{aligned}
0 &\le \Delta_x z - \dot{z} \\
&= 4\zeta^3\Delta_x\zeta q + 12\zeta^2|\nabla\zeta|^2 q + \zeta^4\Delta_x q + 4\zeta^3\langle \nabla\zeta, \nabla q\rangle - \zeta^4\dot{q} - 4\zeta^3\dot{\zeta}q - \mu \\
&\le c\zeta^3|\nabla^2 v| - 2\zeta^4\langle \nabla q, \nabla v\rangle - \zeta^4|\nabla^2 v|^2 + 4\zeta^3\langle \nabla\zeta, \nabla q\rangle - \mu \\
&\le c\zeta^3|\nabla^2 v| + 8\zeta^3 q\langle \nabla\zeta, \nabla v\rangle - \zeta^4|\nabla^2 v|^2 + c\zeta^2|\nabla^2 v||\nabla\zeta|^2 - \mu \\
&\le c\zeta^3|\nabla^2 v| + \epsilon\zeta^4|q||\nabla v|^2 + \frac{\zeta^2|q|}{\epsilon}|\nabla\zeta|^2 - \zeta^4|\nabla^2 v|^2 + c\zeta^2|\nabla^2 v||\nabla\zeta|^2 - \mu \\
&< 0
\end{aligned}
$$

for large μ, which is a contradiction. Hence z remains non-negative and thus

$$
\dot{v} - \frac{1}{2}|\nabla v|^2 = \Delta_x v + \frac{1}{2}|\nabla v|^2 = q(t,x) \ge -\mu t \quad \forall (t,x) \in [t_1,t_2]\times\bar{B}_r(x_0).
$$

Hence (5.7) is established and the proof complete. □

Energy methods

The uniqueness question has already been settled for the Cauchy-Dirichlet problem. Now we want to provide another proof using energy methods, which is interesting by itself. Furthermore it will enable to prove a result about *backwards uniqueness*.

5.2.9 Proposition (Uniqueness revisited). *Let* $n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ *with smooth boundary,* $Q = (0,\infty) \times \Omega$ *and* $u \in C^\infty(\bar{Q})$ *solve*

$$\begin{aligned} \Delta_x u - \dot{u} &= 0 \quad \text{in } Q \\ u &= 0 \quad \text{on } \partial_p Q. \end{aligned}$$

Then $u = 0$.

Proof. Along the heat flow there is a *monotone quantity*[5], namely

$$e(t) = \int_\Omega u(t,x)^2 \, dx.$$

We have

$$\begin{aligned} \dot{e}(t) &= 2\int_\Omega u(t,x)\dot{u}(t,x) \, dx \\ &= 2\int_\Omega u(t,x)\Delta_x u(t,x) \, dx \\ &= -2\int_\Omega |\nabla u(t,x)|^2 \, dx \\ &\le 0. \end{aligned}$$

Hence

$$e(t) \le e(0) = 0$$

and $u = 0$. □

The backwards uniqueness is more subtle.

5.2.10 Proposition (Backward uniqueness). *Let* $n \in \mathbb{N}$, $\Omega \Subset \mathbb{R}^n$ *with smooth boundary,* $T > 0$ $Q = (0,T) \times \Omega$ *and* $u \in C^\infty(\bar{Q})$ *solve*

$$\begin{aligned} \Delta_x u - \dot{u} &= 0 \quad \text{in } Q \\ u(t,\cdot) &= 0 \quad \text{on } \partial\Omega \quad \forall t > 0 \\ u(T,\cdot) &= 0. \end{aligned}$$

Then $u = 0$.

Proof. In addition to Proposition 5.2.9 we calculate

$$\begin{aligned} \ddot{e}(t) &= -4\int_\Omega \langle \nabla u(t,x), \nabla \dot{u}(t,x)\rangle \, dx \\ &= 4\int_\Omega \Delta_x u(t,x)\dot{u}(t,x) \, dx \\ &= 4\int_\Omega (\Delta_x u(t,x))^2 \, dx. \end{aligned}$$

[5] Especially in more complicated problems, such as in fully nonlinear equations, the existence of a monotone quantity often opens the gates to existence and convergence results.

Hence

$$\begin{aligned}\dot{e}(t)^2 &= 4\left(\int_\Omega |\nabla u(t,x)|^2\ dx\right)^2\\ &= 4\left(\int_\Omega u(t,x)\Delta_x(t,x)\ dx\right)^2\\ &\le 4\|u(t,\cdot)\|_{2,\Omega}^2\|\Delta_x u(t,\cdot)\|_{2,\Omega}^2\\ &= e(t)\ddot{e}(t).\end{aligned}$$

Let

$$\tau = \inf\{t > 0 \colon u(s,\cdot) \not\equiv 0 \quad \forall 0 < s < t\} \le T$$

and suppose $\tau > 0$. This implies $e(0) > 0$ and $\dot{e}(0) < 0$. Thus $f(t) = \log e(t)$ satisfies

$$\ddot{f}(t) = \frac{\ddot{e}(t)}{e(t)} - \frac{\dot{e}(t)^2}{e(t)^2} \ge 0 \quad \forall 0 < t < \tau$$

and is hence convex. Thus

$$f((1-s)\tau) \le sf(0) + (1-s)f(\tau) \quad \forall 0 \le s \le 1$$

and

$$e((1-s)\tau) \le e(0)^s e(\tau)^{1-s} = 0,$$

where we used $u(\tau,\cdot) = 0$. This is a contradiction to the definition of τ. □

5.3 Wave equation

In this final section we collect the most basic properties related to the wave equation and its solutions. In order to do so, we first have a look at a first order PDE, the transport equation. We follow Evans' book [1] closely.

Transport equation

5.3.1 Definition. Let $n \in \mathbb{N}$ and $b \in \mathbb{R}^n$. The PDE

$$\dot{u} + \langle b, \nabla u\rangle = 0 \tag{5.8}$$

in $(0,\infty) \times \mathbb{R}^n$ is called the *transport equation.*

5.3.2 Remark. Having a sharp look at (5.8), one can see that a certain directional derivative of u vanishes, namely the one in direction (t,b):

$$\frac{d}{ds}u(t+s, x+sb) = \dot{u} + \langle \nabla u, b\rangle = 0.$$

Hence the value of u is constant along the line

$$\gamma(s) = (t+s, x+sb)$$

and if we know the C^1 initial function

$$g \colon \mathbb{R}^n \to \mathbb{R},$$

we conclude by inserting $s = 0$ and $s = -t$ that

$$u(t, x) = u(0, x - tb) = g(x - tb).$$

If we want to solve the inhomogeneous problem

$$\dot{u} + \langle b, \nabla u \rangle = f,$$

we can proceed similarly and integrate to obtain

$$\begin{aligned} u(t, x) - g(x - tb) &= \int_{-t}^{0} \frac{d}{ds} u(t + s, x + sb) \, ds \\ &= \int_{-t}^{0} f(t + s, x + sb) \, ds \\ &= \int_{0}^{t} f(\tau, x + (\tau - t)b) \, d\tau. \end{aligned}$$

We have proved:

5.3.3 Proposition. *Let $n \in \mathbb{N}$, $g \in C^1(\mathbb{R}^n)$ and $f \in C^1((0, \infty) \times \mathbb{R}^n)$. Then the unique $C^1((0, \infty) \times \mathbb{R}^n)$-solution to the inhomogeneous transport equation with initial values g,*

$$\begin{aligned} \dot{u} + \langle b, \nabla u \rangle = f \quad &\text{in } (0, \infty) \times \mathbb{R}^n \\ u = g \quad &\text{on } \{0\} \times \mathbb{R}^n \end{aligned}$$

is given by

$$u(t, x) = g(x - tb) + \int_{0}^{t} f(\tau, x + (\tau - t)b) \, d\tau.$$

The one-dimensional wave equation

We want to solve the one-dimensional wave equation on the real line, i.e.

$$\begin{aligned} \ddot{u} - u_{,xx} = 0 \quad &\text{in } (0, \infty) \times \mathbb{R} \\ u = g, \dot{u} = h \quad &\text{on } \{t = 0\} \times \mathbb{R}. \end{aligned}$$

Note that we have to give two initial conditions, similar to second order ordinary differential equations. We will heuristically derive a solution. Therefore let

$$v = \dot{u} - u_{,x}.$$

Then

$$\dot{v} + v_{,x} = \ddot{u} + u_{,tx} - u_{,xt} - u_{,xx} = 0.$$

Hence v solves a transport equation and we get

$$v(t, x) = a(x - t),$$

where $a(x) = v(0, x)$. This implies

$$\dot{u} - u_{,x} = a(x - t),$$

which is an inhomogeneous transport equation. Putting $b(x) = u(0, x)$ we get

$$\begin{aligned} u(t, x) &= b(x + t) + \int_0^t a(x + (t - s) - s) \; ds \\ &= b(x + t) + \frac{1}{2} \int_{x-t}^{x+t} a(y) \; dy. \end{aligned}$$

Now we insert the initial conditions:

$$a(x) = v(0, x) = u_{,t}(0, x) - u_{,x}(0, x) = h(x) - g'(x).$$

Inserting gives

$$\begin{aligned} u(t, x) &= g(x + t) + \frac{1}{2} \int_{x-t}^{x+t} (h(y) - g'(y) \; dy \\ &= \frac{1}{2}(g(x + t) + g(x - t)) + \frac{1}{2} \int_{x-t}^{x+t} h(y) \; dy. \end{aligned}$$

We obtain

5.3.4 Theorem (D'Alembert formula). *Let $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$. Then*

$$u(t, x) = \frac{1}{2}(g(x + t) + g(x - t)) + \frac{1}{2} \int_{x-t}^{x+t} h(y) \; dy$$

is a solution to the wave equation and

$$\lim_{(t,x) \to (0,x_0)} u(t, x) = g(x_0), \qquad \lim_{(t,x) \to (0,x_0)} \dot{u}(t, x) = h(x_0).$$

Bibliography

[1] Lawrence C. Evans, *Partial differential equations*, 1. ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, 1998.

[2] Avner Friedman, *Partial differential equations of parabolic type*, Dover Publications, Mineola, N.Y., 2008.

[3] Claus Gerhardt, *Analysis I*, Verlag Claus Gerhardt, Heidelberg, 2002.

[4] ______, *Curvature problems*, Series in Geometry and Topology, vol. 39, International Press of Boston Inc., Sommerville, 2006.

[5] David Gilbarg and Neil Trudinger, *Elliptic partial differential equations of second order*, reprint of the 1998 edition ed., Classics in Mathematics, Springer, 2001.

[6] Jacques Hadamard, *Extension a l'équation de la chaleur d'un théoreme de a. harnack*, Rend. Circ. Mat. Palermo Ser. 2 **3** (1954), no. 3, 337–346.

[7] Eberhard Hopf, *A remark on linear elliptic differential equations of second order*, Proc. Am. Math. Soc. **3** (1952), no. 5, 791–793.

[8] Nicolai Krylov, *Lectures on elliptic and parabolic equations in Hölder spaces*, Graduate Studies in Mathematics, vol. 12, American Mathematical Society, 1996.

[9] Ernst Kuwert, *Analysis II*, Lecture notes, 2016.

[10] Peter Li and Shing-Tung Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), no. 1, 153–201.

[11] Gary Lieberman, *Second order parabolic differential equations*, World Scientific, Singapore, 1998.

[12] Norman Meyers and James Serrin, $H = W$, Proc. Natl. Acad. Sci. USA **51** (1964), no. 6, 1055–1056.

[13] Oskar Perron, *Eine neue Behandlung der ersten Randwertaufgabe für* $\Delta u = 0$, Math. Z. **18** (1923), no. 1, 42–54.

[14] Bruno Pini, *Sulla soluzione generalizzata di Wiener per il primo problema di valori al contorno nel caso parabolico*, Rend. Sem. Mat. Univ. Padova **23** (1954), 422–434.

[15] Bernhard Riemann, *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Größe*, Ph.D. thesis, Göttingen, 1851.

[16] Ben Schweizer, *Partielle Differentialgleichungen*, Masterclass, Springer Spektrum, Berlin, Heidelberg, 2013.

[17] William Ziemer, *Weakly differentiable functions*, Graduate Texts in Mathematics, no. 120, Springer, New York, 1989.