DRAFT PARABOLIC PDE NOTES

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Large disclaimer:

These are badly written draft lecture notes which contain numerous errors, a few of which I know about and most of which I do not. These should therefore only be used during the workshop – later, I will upload a (somewhat) corrected version.

Please help me in improving these by sending me corrections of the numerous mistakes to

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I would also be grateful for any comments/advice on anything that is "technically correct but unclear".

Due to their "extreme draft" nature, I would be grateful if you would not share these with anyone outside of the conference before I write some corrections!

1. Parabolic PDEs and function spaces

1.1. **Degrees of nonlinearity.** The model parabolic PDE is the heat equation

$$u_t = \Delta u \ ,$$

which takes a temperature distribution and then smooths it out over time. More accurately, for this to be well posed (unique solution depending continuously on initial data), we could consider the Dirichlet problem for the heat equation: For some $T > 0$ and some domain $\Omega \subset \mathbb{R}^n$, we consider $u : \Omega \times [0, T) \to \mathbb{R}$ with initial data $u_0 : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} u_t - \Delta u = 0 & \text{for } (x,t) \in \Omega \times [0,T) \\ u(x,0) = u_0 & \text{for } x \in \Omega \\ u(x,t) = u_0 & \text{for } (x,t) \in \partial\Omega \times [0,T) \ . \end{cases}$$

It will be useful at this point to define $\Omega_T = \Omega \times [0, T)$.

The above equation is well posed and satisfies good properties (see later sections of the notes). An alternative to Dirichlet boundary conditions could be Neumann boundary conditions, in which the last line is replaced by

$$\gamma \cdot Du(x,t) + e(x,t)u = h(x,t) \qquad \text{for } (x,t) \in \partial\Omega \times [0,T)$$

for some functions e, f on $\partial\Omega \times [0, T)$ and γ a vector field on $\mathbb{R}^n$ such that $n \cdot \gamma > b > 0$ where n is the normal to $\partial\Omega$.

For this course, to avoid dealing with boundary values and compactness issues, we will be considering $u : M^n \times [0, T) \to \mathbb{R}$ for some compact M^n, which will typically be S^n (if you are not used to manifolds, imagine this to be functions on the n torus, which is equivalent to considering n-periodic functions on $\mathbb{R}^n$ – these functions are entirely defined by their values on the unit cube). For such a u we will consider

$$\begin{cases} u_t - \Delta u = 0 & \text{for } (x,t) \in M^n \times [0,T) \\ u(x,0) = u_0 & \text{for } x \in M^n \end{cases}$$

We will say that a matrix a^{ij} is λ-Λ positive definite for some $0 < \lambda < \Lambda$ if for any $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\lambda|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \ . \tag{1}$$

More generally we need to consider more complicated PDEs than the heat equation, in each case replacing the first equation in the above equations:

Linear parabolic PDEs: We consider the operator

$$Lu := u_t - a^{ij}(x,t)D^2_{ij}u - b^i(x,t)D_iu - c(x,t)u$$

where L is said to be *parabolic* (or λ-Λ parabolic) if there exists $0 < \lambda < \Lambda$ so that for any $(x, t) \in \Omega_T$, $a^{ij}(x, t)$ is λ-Λ positive definite (in the sense of (1)). We consider the equation

$$Lu = f$$

on Ω_T. The exact assumptions on the regularity of the coefficients a^{ij}, b^i, c, f will turn out to be very important later. The theory of linear PDEs is vital to us for nonlinear PDEs, particularly as these are linearisations of geometric PDEs.

Quasilinear equations: We could consider instead

$$Q(u) := u_t - A^{ij}(Du, u, x, t)D^2_{ij}u - B(Du, u, x, t) \ ,$$

where we would replace the heat equation with $Q(u) = 0$ (note that every linear equation is a quasi-linear equation!). This equation will be called uniformly parabolic at u there exists $0 < \lambda < \Lambda$ such that $A^{ij}|_{(x,t,u,Du)}$ is λ-Λ positive definite. Equations of this form include mean curvature equations, such as graphical mean curvature flow given by

$$u_t = \left(\delta^{ij} - \frac{D^i u D^j u}{1 + |Du|^2}\right) D^2_{ij}u \ .$$

Fully nonlinear parabolic PDEs: Here we consider

$$Pu := u_t - F(D^2u, Du, u, x, t)$$

which will be called parabolic at u if its linearisation is a parabolic operator, and we would aim to solve $Pu = 0$. Assuming that F is C^1 in every entry and writing $F(r, p, z, x, t)$ for $r \in \mathbb{R}^{n\times n}$, $p \in \mathbb{R}^n$ and $z \in \mathbb{R}$, then the above equation is parabolic at u iff its linearisation is parabolic at u, that is

$$\begin{aligned} L_{P,u}w &:= \frac{d}{ds}\Big|_{s=0} \left((u + sw)_t - F(x, t, u + sw, Du + sDw, D^2u + sD^2w)\right) \\ &= w_t - \frac{\partial F}{\partial r_{ij}}\Big|_{(D^2u,Du,u,x,t)} D^2_{ij}w - \frac{\partial F}{\partial p_i}\Big|_{(D^2u,Du,u,x,t)} D_i w - \frac{\partial F}{\partial z}\Big|_{(D^2u,Du,u,x,t)} w \end{aligned} \tag{2}$$

is also parabolic. Equivalently there exists a $0 < \lambda < \Lambda$ such that $\frac{\partial F}{\partial r_{ij}}\big|_{(x,t,u,Du,D^2u)}$ is λ-Λ positive definite. Examples of such PDEs include the parabolic Monge-Ampere equation

$$u_t = \det(D^2u)$$

for convex u, or the Gauss curvature flow

$$u_t = \frac{\det(D^2u)}{(1 + |Du|)^{\frac{n+1}{2}}}$$

As we will need them regularly, we introduce the notation

$$\frac{\partial F}{\partial r_{ij}}\Big|_{(D^2u,Du,u,x,t)} = F^{ij}|_u, \qquad \frac{\partial F}{\partial p_i}\Big|_{(D^2u,Du,u,x,t)} = F_{p^i}\big|_u, \qquad \frac{\partial F}{\partial z}\Big|_{(D^2u,Du,u,x,t)} = F_z\big|_u \ .$$

Danger: Despite only u being mentioned in the restriction, this actually depends on up to the second derivatives of u. I will also abuse this notation further, for example writing $F^{ij}|_{u(x,t)}$ for the function $F^{ij}|_u$ at the point (x, t). The restriction will also be dropped if it is clear from context.

Usually in nonlinear PDEs as above, we will not be able to prove uniform parabolicity for all u. Functions u such that make the PDE parabolic are called *admissable*.

Definition 1 (Admissability)**.** We define the set of all admissable functions with parabolicity constants λ-Λ to be

$$\Gamma_{\lambda,\Lambda}(\Omega_T) := \{u \in C^{2;1}(\overline{\Omega_T}) : L_{p,u} \text{ is } \lambda\text{-}\Lambda \text{ parabolic}\}\ ,$$

where $C^{2;1}(\overline{\Omega_T})$ is the set of all functions twice continuously differentiable in space, once continuously differentiable in time. We will say that initial data $u_0 : \Omega \to \mathbb{R}$ is λ-Λ admissable if there is a time $\tau > 0$ so that $\tilde{u}_0(x,t) := u_0(x)$ has $\tilde{u}_0 \in \Gamma_{\lambda,\Lambda}(\Omega_\tau)$.

Note that if $F^{ij}|_{u_0}|_{t=0}$ is λ-Λ positive definite, then by continuity, u_0 will be $\frac{\lambda}{2}$-2Λ admissable (or indeed $\frac{\lambda}{\mu}$-$\mu\Lambda$ admissable for any $\mu > 1$). In general we will almost always need to assume some convexity of $\Gamma_{\lambda,\Lambda}$

Example 1. The Monge–Ampere equation above we have $F = \det(D^2u)$ so

$$\frac{\partial F}{\partial r_{ij}} = [\mathrm{adj}(D^2u)]_{ij} = [(D^2u)^{-1}]_{ij} \det(D^2u)\ .$$

This is clearly only positive definite if D^2u is positive definite, i.e. admissability will require that u is strictly convex.

1.2. **Parabolic function spaces.** We define parabolic Hölder norms on the basis that for the heat flow one time derivative corresponds to two space derivatives. We define parabolic function spaces and distances to respect this: For a domain $\Omega \subset \mathbb{R}^n$, we define

$$\Omega_T := \Omega \times [0,T)\ ,$$

and define a metric on Ω_T by

$$d((x,t),(y,s)) = \max\{|x-y|, |t-s|^{\frac{1}{2}}\}\ .$$

A parabolic cylinder of radius r is given by

$$P(x,t,r) = B_r(x) \times (t-r^2, t] = \{(y,s) \in \Omega_T : d((x,t),(y,s)) < r \text{ and } s \leq t\}\ .$$

The parabolic boundary of a set $S \subseteq \mathbb{R}^n \times \mathbb{R}$, written $\mathcal{P}(\Omega_T)$ is the set of all points such that any parabolic cylinder contains points outside the set. In particular

$$\mathcal{P}(\Omega_T) = (\partial\Omega \times [0,T)) \cup (\Omega \times \{0\})$$

We define the parabolic Hölder semi-norm on Ω_T to be

$$[u]_{\alpha,\Omega_T} = \sup_{(x,t),(y,s)\in\Omega_T} \frac{|u(x,t)-u(y,s)|}{d((x,t),(y,s))^\alpha}\ .$$

Unfortunately, for a complete definition, we also need a further semi-norm, just in time. For $u : \Omega_T \to \mathbb{R}$ we define

$$\langle u\rangle_\alpha = \sup_{x\in\Omega} \sup_{\substack{t,s\in[0,T)\\ t\neq s}} \frac{|u(x,t)-u(x,s)|}{|t-s|^\alpha}\ .$$

From these semi-norms we can define the parabolic Hölder norms. A multi-index is an n-tuple, $\beta = (\beta^1, \ldots, \beta^n) \in \mathbb{Z}^n_{\geq 0}$, where we define $|\beta| = \beta^1 + \ldots + \beta^n$. Given any such β we define $D_\beta u = D_1^{\beta_1} D_2^{\beta_2} \ldots D_n^{\beta_n} u$.

For a function $u : \Omega_T \to \mathbb{R}$, $k \in \mathbb{N}$ and $\alpha \in (0,1)$,

$$
\begin{aligned}
|u|_{0,\Omega_T} = |u|_{C^0(\Omega_T)} &= \sup_{(x,t)\in\Omega_T} |u(x,t)| \\
|u|_{k,\Omega_T} = |u|_{C^{k;\frac{k}{2}}(\Omega_T)} &= \sum_{2r+|\beta|\leq k} |D_t^r D_\beta u(x,t)|_{0,\Omega_T} \\
[D^{k,\frac{k}{2}} u]_{\alpha,\Omega_T} &= \sum_{2r+|\beta|=k} [D_t^r D_\beta u(x,t)]_{\alpha,\Omega_T} \\
\langle D^{k-1,\frac{k-1}{2}} u\rangle_{\frac{\alpha+1}{2},\Omega_T} &= \sum_{2r+|\beta|=k-1} \langle D_t^r D_\beta u(x,t)\rangle_{\frac{1+\alpha}{2},\Omega_T} \\
|u|_{k+\alpha,\Omega_T} = |u|_{C^{k+\alpha;\frac{k+\alpha}{2}}(\Omega_T)} &= |u|_{k,\Omega_T} + [D^{k,\frac{k}{2}} u]_{\alpha,\Omega_T} + \langle D^{k-1,\frac{k-1}{2}} u\rangle_{\frac{1+\alpha}{2},\Omega_T}
\end{aligned}
$$

where in the above. Importantly, for $k \neq 1$, the $\langle D^{k-1,\frac{k-1}{2}} u\rangle_{\frac{1+\alpha}{2},\Omega_T}$ term essentially doesn't contribute to the Hölder norm (also note that this is zero for $k = 0$) in the sense that we could remove this and get another equivalent norm - we will see this in our interpolation lemma later. This term is an irritating necessity for $k = 1$ (see Exercise 6)! We will define function spaces by those functions which have finite norm, so,

$$
C^{k+\alpha;\frac{k+\alpha}{2}}(\Omega_T) = \{u \in C^0(\Omega_T) : |u|_{C^{k+\alpha;\frac{k+\alpha}{2}}(\Omega_T)} < \infty\} .
$$

Suppose that Ω is a domain of compact closure and $T \in \mathbb{R}_{>0}$. We begin by stating a number of properties of $C^{k+\alpha;\frac{k+\alpha}{2}}(\Omega_T)$:

- We may obtain elliptic Hölder norms for a function $f : \Omega \to \mathbb{R}$, by extending this to be constant in time on Ω_T, i.e. $f_T(x,t) = f(x)$. Then $|f|_{C^{k,\alpha}(\Omega)} = |f_T|_{k+\alpha,\Omega_T}$. As we will mainly be applying this on initial data, from now on we will denote the elliptic Hölder norm by $|f|^0_{k+\alpha} := |f|_{C^{k,\alpha}(\Omega)} = |f_T|_{k+\alpha,\Omega_T}$.
- Suppose that $u \in C^{k+\alpha;\frac{k+\alpha}{2}}(\Omega_T)$ where Ω_T is a set of compact closure. Then there exists a unique continuous extension of u to $\overline{\Omega_T}$, such that all derivatives are continuous up to the boundary.
- The Extension Property: Suppose that $u \in C^{k+\alpha;\frac{k+\alpha}{2}}(\Omega_T)$ where Ω_T is a set of compact closure and $\partial\Omega$ is smooth (in fact, it is enough for it to be written locally as a $C^{2,\alpha}$ graph). Suppose additionally that $\overline{\Omega} \subset \Omega' \subset \mathbb{R}^{n+1}$ for some open Ω'. Then there exists a $C = C(\Omega,\Omega') > 0$ such that any $u \in C^{2+\alpha;\frac{2+\alpha}{2}}(\Omega_T)$ may be extended to $\tilde{u} : \Omega' \times (-1, T+1)$ where $\tilde{u} = 0$ in a neighbourhood of $\partial(\Omega' \times (-1, T+1))$ and

  $$|\tilde{u}|_{k+\alpha,\Omega'\times(-1,T+1)} \leq C|u|_{k+\alpha,\Omega_T}$$

  (see Gilbarg and Trudinger [3, Section 6.9 on p136] for the elliptic equivalent of this).
- $C^{k+\alpha;\frac{k+\alpha}{2}}(\Omega_T)$ is a Banach space.
- If $k, l \in \mathbb{Z}_{\geq 0}$ and $\alpha, \beta \in (0,1)$ with $l + \beta < k + \alpha$ then

  $$C^{k+\alpha;\frac{k+\alpha}{2}}(\Omega_T) \subset C^{l+\beta;\frac{l+\beta}{2}}(\Omega_T) .$$

  Furthermore, using Arzèla-Ascoli the inclusion map in the above is a compact mapping – that is to say, any bounded set is $C^{k+\alpha;\frac{k+\alpha}{2}}(\Omega_T)$ is a set of compact closure in

$C^{l+\beta;\frac{l+\beta}{2}}(\Omega_T)$. For example, for $k = l = 0$ we have that if for some sequence $u_i \in C^{\alpha;\frac{\alpha}{2}}$ with $|u_i|_\alpha < C$:

(1) The sequence u_i is equicontinuous (by definition of the Hölder norm) so by Arzela-Ascoli (Theorem 33), there is a subsequence (also labelled with i) which converges to some continuous u_∞.
(2) In fact, $|u_\infty|_\alpha \leq C$ (exercise)
(3) Furthermore, $|u_i - u_\infty|_\beta \to 0$ as $i \to \infty$ (exercise).
(4) More generally this is easily extended to $k, l \neq 0$. (another exercise)

Assumption 1. From now on assume that Ω is of compact closure!

Example 2. If $u \in C^{2+\alpha;\frac{2+\alpha}{2}}(\Omega_T)$ then in terms of elliptic Hölder spaces,

$$u, Du, u_t, D^2u \in C^0(\Omega_T)$$

additionally that for all $t \in [0, T)$,

$$\dot{u}(\cdot, t), D^2u(\cdot, t) \in C^\alpha(\Omega) \ .$$

and for any $x \in \Omega$,

$$\dot{u}(x, \cdot), D^2u(x, \cdot) \in C^{\frac{\alpha}{2}}([0, T)), \qquad Du(x, \cdot) \in C^{\frac{1+\alpha}{2}}([0, T)) \ .$$

1.3. **Strategy for fully nonlinear parabolic PDE long time existence theorems.** Firstly, not all parabolic PDEs have smooth solutions for all time (e.g. consider the famous singularities of mean curvature flow or Ricci flow)! In these cases, it is often point (3) below which generally fails in one way or the other. Suppose now that that F is smooth and "sufficiently nice" (see section 3.2). Our general strategy for proving that a geometric parabolic PDE exists for all time is below:

(1) Short time existence (under fairly weak conditions on F, if the initial data u_0 is admissable, the equation always exist for a short time in $C^{2+\alpha;\frac{2+\alpha}{2}}$).
(2) Boot-strapping estimates bumps up the regularity from $2+\alpha$ to $k+\alpha$ for arbitrarily large k. This gives uniform estimates while the flow is uniformly bounded in $C^{2+\alpha;\frac{2+\alpha}{2}}$ and it remains uniformly parabolic (i.e. λ doesn't decay to zero, Λ doesn't explode to infinity).
(3) Find some way (typically by the maximum principle) to show that while the flow exists, it remains uniformly parabolic and has a uniform $C^{2;1}$ bound. This part of the process necessarily is not always possible and typically uses the structure of the specific equation in question.
(4) Use Krylov-Safonov estimates to prove that once you have uniform $C^{2;1}$ estimates, you have uniform $C^{2+\alpha;\frac{2+\alpha}{2}}$ estimates. (For this, you will need to assume at least C^4 initial data).
(5) Therefore: Your PDE has a solution for some small time. For a contradiction, we now suppose that $T < \infty$ is the maximum time at which there exists a solution. Up to time T the above implies that the solution is uniformly bounded in $C^{2+\alpha;\frac{2+\alpha}{2}}$ and so by boot strapping, the solution is smooth. Taking a limit of $u(\cdot, t)$ as $t \to T$ we get a smooth function u_T which is admissable. We may now restart the flow using our short time existence theorem extending the solution up to time $T + \epsilon$ and contradicting the maximality of T.

If our flow is only quasilinear, then everything happens one order lower. Boot strapping works for $C^{1+\alpha;\frac{1+\alpha}{2}}$ functions. We only need to get C^1 estimates and parabolicity by hand (replacing part 3), and part 4 can be replaced by the Nash–Moser–De Giorgi estimates.

1.4. **Equivalent norms, Ehrling's Lemma and interpolation.** Our next lemma points out that (up to a constant) we could drop the final term of the definition of parabolic Hölder norm as long as $k \neq 1$. We prove this for $k = 2$, but the general case is identical.

Lemma 1 (Dropping the $\langle\cdot\rangle$ term)**.** *Suppose that $\Omega \subset \Omega'$ such that there is a $\lambda \leq 1$ so that for all $x \in \Omega$, $B_\lambda(x) \in \Omega'$. Then for $P = \Omega' \times [-\lambda^2, T + \lambda^2]$, there is a constant $C = C(n, \lambda)$ such that for any $u \in C^{2+\alpha;\frac{2+\alpha}{2}}(P)$ and $1 \leq i \leq n$,*

$$\langle D_i u\rangle_{\frac{1+\alpha}{2},\Omega_T} \leq [D^{2,1}u]_{\alpha,P} + C|Du|_{0,P}\ . \tag{3}$$

Proof. For any $x \in \Omega$, we estimate

$$U(x) = \sup_{t,s\in[0,T),t\neq s} \frac{|D_i u(x,t) - D_i u(x,s)|}{|t-s|^{\frac{1+\alpha}{2}}}$$

directly.

Suppose first that $|t - s| > \lambda$. Then $U(x) < 2\lambda^{-1}|Du|$ and we are done.

Now suppose that $|t-s| < \lambda$ and (wlog) $t > s$. Recall that the integral form of the Taylor expansion states that

$$u(x + de_i, t) = u(x) + dD_i u(x) + \int_0^d (d - z)D_{ii}u(x + z)dz$$

Then, writing $d = \sqrt{t-s}$ we may estimate

$$\begin{aligned}
|D_i u(x,t) - D_i u(x,s)| &= |D_i u(x,t) - d^{-1}(u(x+de_i,t) - u(x,t)) \\
&\quad + d^{-1}(u(x+de_i,t) - u(x,t)) - d^{-1}(u(x+de_i,s) - u(x,s)) \\
&\quad + d^{-1}(u(x+de_i,s) - u(x,s)) - D_i u(x,s)| \\
&= \Bigg| -\frac{1}{d}\int_0^d (d-z)D_{ii}u(x+z,t)dz \\
&\quad + d^{-1}(u(x+de_i,t) - u(x+de_i,s)) - d^{-1}(u(x,t) - u(x,s)) \\
&\quad + \frac{1}{d}\int_0^d (d-z)D_{ii}u(x+z,s)dz \Bigg| \\
&= \Bigg| -\frac{1}{d}\int_0^d (d-z)(D_{ii}u(x+z,s) - D_{ii}u(x+z,t))dz \\
&\quad + d^{-1}\int_0^{d^2} (u_t(x+de_i,s+z) - u_t(x,s+z))dz \Bigg| \\
&\leq \left| \frac{[D_{ii}u]_\alpha}{d} \int_0^d (d-z)d^\alpha dz + d^{-1}[u_t]_{\alpha,P}\int_0^{d^2} d^\alpha dz \right| \\
&\leq d^{1+\alpha}\left(\frac{1}{2}[D_{ii}u]_{\alpha,P} + [u_t]_{\alpha,P}\right)
\end{aligned}$$

□

Corollary 2. *If Ω is a compact domain with a $C^{k+\alpha}$ boundary. Then there exists a constant C depending only on n and Ω so that*

$$\langle D_i u\rangle_{\frac{1+\alpha}{2},\Omega_T} \leq [D^{2,1}u]_{\alpha,\Omega_T} + C|Du|_{0,\Omega_T}$$

This follows from the extension property from the previous subsection.

Corollary 3. *Let $k \in \mathbb{Z}_{\geq 0}$, $k \neq 1$ and $\alpha \in (0,1)$. Let Ω be a compact domain with a $C^{k+\alpha}$ boundary. Then there exists a constant C depending only on n and Ω so that*

$$\langle D^{k-1,\frac{k-1}{2}}u\rangle_{\frac{\alpha+1}{2},\Omega_T} \leq C([D^{k,\frac{k}{2}}u]_{\alpha,\Omega_T} + |D^{k-1,\frac{k-1}{2}}u|_{0,\Omega_T})$$

Sketch of proof. This is identical to the above. □

The following is a very useful lemma which is applicable in a wide number of places.

Lemma 4 (Ehrling's Lemma)**.** *Let U, V, W be Banach spaces such that U is continuously embedded in V which is continuously embedded in W i.e.*

$$U \to V \to W\ .$$

Suppose that $U \to V$ is compact. Then for all $\epsilon > 0$ there exists a constant $c_\epsilon = c_\epsilon(\epsilon)$ such that

$$\|u\|_V \leq \epsilon\|u\|_U + c_\epsilon\|u\|_W\ .$$

Proof. Suppose not. Then there is an $\epsilon > 0$ and a sequence $u_k \in U$ such that

$$\|u_k\|_V > \epsilon\|u_k\|_U + k\|u_k\|_W\ .$$

Wlog (by multiplying through by a constant) we may take $\|u\|_V = 1$. This inequality implies that $\|u_k\|_W \to 0$ in W and $\|u_k\|_U \leq \epsilon^{-1}$. By compactness of the first embedding there exists a subsequence (also called u_k) such that $u_k \to u$ for some $u \in V$. We also have that $\|u\|_V = 1$ by continuity of the norm. By continuity of the second embedding we also have that u_k converges to u in the $\|\cdot\|_W$ norm. We also have that $u_k \to 0 = u$. By injectivity of the second mapping (from the embedding property) $u = 0$ in V, a contradiction as $\|u\|_V = 1$. □

In particular, note that for Ω of compact closure, for $l + \beta < k + \alpha$, for $k, l \in \mathbb{Z}_{\geq 0}$ and $\alpha, \beta \in (0,1)$, then $C^{k+\alpha;\frac{k+\alpha}{2}}(\overline{\Omega_T}) \subset C^{l+\beta;\frac{l+\beta}{2}}(\overline{\Omega_T}) \subset C^0(\overline{\Omega_T})$ where the first inclusion is compact by Arzelà–Ascoli as mentioned in the properties above. The disadvantage of the above is that we have no idea of how c_ϵ depends on ϵ (and cannot have any idea).

For Hölder norms we can do better than this, namely the following:

Proposition 5 (Hölder interpolation)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ has compact closure and a smooth boundary (in fact a boundary that can be written as a $C^{k+\alpha}$ graph is enough). Let $k \in \mathbb{Z}_{\geq 0}$ and $\alpha \in (0,1)$. Then there exists a constant $C = C(k,\alpha)$ such for any $u \in C^{k+\alpha;\frac{k+\alpha}{2}}$ and any $l \in \mathbb{Z}_{\geq 0}$ and $\beta \in (0,1)$ with $l + \beta < k + \alpha$,*

$$|u|_{l+\beta} \leq C|u|_0^{\frac{(k+\alpha)-(l+\beta)}{k+\alpha}}|u|_{k+\alpha}^{\frac{l+\beta}{k+\alpha}}$$

Below I sketch out a few points on how this can be proven where (for simplicity) I take $k = 2$. The general case follows similarly.

Lemma 6 (Interpolation)**.** *Let P be one of either $\mathbb{R}^n \times [0,\infty)$ or $\mathbb{R}^n \times (-\infty, 0]$. Then there is a constant $C = C(n)$ such that for any $u \in C^{2+\alpha;\frac{2+\alpha}{2}}(P)$ and $1 \le i,j \le n$,*

$$|u_t|_{0,P} \le \epsilon[D^{2,1}u]_{\alpha,P} + C\epsilon^{-\frac{2}{\alpha}}|u|_{0,P} \tag{4}$$

$$|D^2_{ij}u|_{0,P} \le \epsilon[D^{2,1}u]_{\alpha,P} + C\epsilon^{-\frac{2}{\alpha}}|u|_{0,P} \tag{5}$$

$$|D_i u|_{0,P} \le \epsilon[D^{2,1}u]_{\alpha,P} + C\epsilon^{-\frac{1}{1+\alpha}}|u|_{0,P} \tag{6}$$

$$[D_i u]_{\alpha,P} \le \epsilon[D^{2,1}u]_{\alpha,P} + C\epsilon^{-(1+\alpha)}|u|_{0,P} \tag{7}$$

$$[u]_{\alpha,P} \le \epsilon[D^{2,1}u]_{\alpha;P} + C\epsilon^{-\frac{\alpha}{2}}|u|_{0,P} \tag{8}$$

$$\langle D_i u\rangle_{\alpha} \le \epsilon[D^{2,1}u]_{\alpha;P} + C\epsilon^{-(1+\alpha)}|u|_{0,P} \tag{9}$$

Proof. Our aim will be to prove this for $\epsilon = 1$. Once we have this, the general ϵ case follows by scaling.

For example, suppose that we know that there is a C so that for any $f \in C^{2+\alpha;\frac{2+\alpha}{2}}(P)$, the first of these holds, that is

$$|f_t|_{0,P} \le [D^{2,1}f]_{\alpha,P} + C|f|_{0,P} \ .$$

We consider the parabolic rescaling of u: For any $\lambda \in \mathbb{R}_+$, define

$$u^\lambda(x,t) = u(\lambda x, \lambda^2 t) \ .$$

Then as $|u^\lambda_t|_{0,P} = \lambda^2|u_t|_{0,P}$, $[D^{2,1}u^\lambda]_{\alpha,P} = \lambda^{2+\alpha}[D^{2,1}u]_{\alpha,P}$, $|u^\lambda|_{0,P} = |u|_{0,P}$, by applying the above with $f = u^\lambda$,

$$\lambda^2|u_t|_{0,P} \le \lambda^{2+\alpha}[D^{2,1}u]_{\alpha,P} + C|u|_{0,P} \ .$$

Equivalently

$$|u_t|_{0,P} \le \lambda^{\alpha}[D^{2,1}u]_{\alpha,P} + \lambda^{-2}C|u|_{0,P} \ .$$

so setting $\lambda = \epsilon^{\frac{1}{\alpha}}$ gives the first equation.

We now prove the first of the above. We have that

$$|u_t(x,t)| \le |u_t(x,t) - (u(x,t+1) - u(x,t))| + 2|u|_{0,P} = |u_t(x,t) - u_t(x,\theta)| + 2|u|_{0,P} \le [u_t]_{\alpha,P} + 2|u|_{0,P}$$

therefore we have the first above. I leave the others as an exercise, but note that this is also in either the notes of Picard [7, Proposition 3, p12] or a book by Krylov [4, Theorem 8.8.1, p124]. □

Corollary 7. *If Ω is a compact domain with a $C^{2+\alpha}$ boundary. Then there exists a constant C depending only on n and Ω so that all of the estimates in Lemma 6 hold.*

Proof. This follows from the extension property for Hölder spaces. □

Corollary 8. *For any $k \ge 2$ and any compact domain Ω with a smooth boundary ($C^{k+\alpha}$ boundary), an equivalent norm to $|u|_{k+\alpha,\Omega_T}$ is*

$$|u|'_{k+\alpha,\Omega_T} = |u|_0 + [D^{k,\frac{k}{2}}u]_\alpha \ .$$

Corollary 9. *If Ω is a compact domain with a $C^{2+\alpha}$ boundary. Then there exists a constant C depending only on n and Ω so that*

$$|D^2_{ij}u|_{0,\Omega_T} + |u_t|_{0,\Omega_T} \leq C|u|_{0,\Omega_T}^{\frac{\alpha}{2+\alpha}}[D^{2,1}u]_{\alpha,\Omega_T}^{\frac{2}{2+\alpha}} \tag{10}$$

$$|D_iu|_{0,\Omega_T} \leq C|u|_{0,\Omega_T}^{\frac{1+\alpha}{2+\alpha}}[D^{2,1}u]_{\alpha,\Omega_T}^{\frac{1}{2+\alpha}} \tag{11}$$

$$[D_iu]_{\alpha,\Omega_T} \leq C|u|_{0,\Omega_T}^{\frac{1}{2+\alpha}}[D^{2,1}u]_{\alpha,\Omega_T}^{\frac{1+\alpha}{2+\alpha}} \tag{12}$$

$$[u]_{\alpha,\Omega_T} \leq C|u|_{0,\Omega_T}^{\frac{2}{2+\alpha}}[D^{2,1}u]_{\alpha,\Omega_T}^{\frac{\alpha}{2+\alpha}} \tag{13}$$

Proof. This follows from choosing the "right" ϵ in Lemma 6. For example, from the second inequality, we may set $\epsilon = |u|_0^{\frac{\alpha}{2+\alpha}}[D^{2,1}u]_\alpha^{-\frac{\alpha}{2+\alpha}}$ to get the first part of the first equation. □

Lemma 10. *For any $0 < \beta < \alpha < 1$, if $u \in C^{\alpha;\frac{\alpha}{2}}(\overline{\Omega}_T)$ then $[u]_{\beta,\Omega} \leq 2[u]_{\alpha,\Omega}^{\frac{\beta}{\alpha}} \operatorname{osc}(u)^{1-\frac{\beta}{\alpha}}$.*

Proof. We have that

$$\frac{|u(x,t)-u(y,s)|}{d((x,t),(y,s))^\beta} = \left(\frac{|u(x,t)-u(y,s)|}{d((x,t),(y,s))^\alpha}\right)^{\frac{\beta}{\alpha}} |u(x,t)-u(y,s)|^{1-\frac{\beta}{\alpha}} \leq [u]_\alpha^{\frac{\beta}{\alpha}} \operatorname{osc}(u)^{1-\frac{\beta}{\alpha}} .$$

Taking a supremum gives the statement. □

Corollary 11. *If Ω is a compact domain with a $C^{2+\alpha}$ boundary. Then there exists a constant C depending only on n and Ω so that for all $l \in \{0,1,2\}$, $0 < \beta < \alpha < 1$,*

$$|u|_{l+\beta,\Omega_T} \leq C|u|_{0,\Omega_T}^{1-\frac{l+\beta}{2+\alpha}}|u|_{2+\alpha,\Omega_T}^{\frac{l+\beta}{2+\alpha}}$$

Proof. This follows by going through the cases and applying a combination of Corollary 9 and Lemma 10. □

We give a final useful Lemma for computing Hölder exponents:

Lemma 12. *Suppose that $f, g \in C^{\alpha;\frac{\alpha}{2}}(\Omega_T)$ then*

$$[f \cdot g]_\alpha \leq |f|_0[g]_\alpha + |g|_0[f]_\alpha .$$

If additionally, $\Phi : \mathbb{R} \to \mathbb{R}$ is Lipschitz then

$$[\Phi(f)]_\alpha \leq |\Phi|_{\mathrm{Lip}}[f]_\alpha .$$

More generally, if $\Phi : \mathbb{R} \to \mathbb{R}$ is C^β then

$$[\Phi(f)]_{\beta\alpha} \leq C[\Phi]_\beta[f]_\alpha .$$

The proof is left as an exercise.

1.5. **Hölder spaces on compact manifolds.** Suppose now that M is a smooth compact manifold. Then there exists a finite atlas of charts, given by open sets $V_i \subset M$ and mappings $\varphi_i : M \to B_1(0) \subset \mathbb{R}^n$ for $1 \leq i \leq N$. For example, at any point p in M geodesic normal coordinates at gives such a mapping at the point for balls of sufficiently small radius. These then form an open cover, and the required set a finite subcover.

We may now define Hölder norms on $M_T = M \times [0,T)$ by

$$|u|_{k+\alpha} = \sum_{i=1}^N |u \circ \phi_i^{-1}|_{k+\alpha, B_1(0)\times[0,T)} .$$

We may now define parabolic Hölder spaces exactly as before (and again, this forms a nice Banach space). All interpolation theorems hold as above (with slightly different constants depending on the open cover). Our definition depends on the choice of open cover, but this will not be important for the applications we have in mind.

1.6. **Some exercises on Hölder spaces.** As there were a large number of unproven results in this section, I have collected most of these as the exercises below to have a go at/think about.

Exercise 1. We define parabolic dilation $D_\lambda(x,t) = (\lambda x, \lambda^2 t)$, and we may define a blowup on $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ (at 0) by $u^\lambda := \lambda^{-1} u \circ D_\lambda$. What effect does D_λ have on the parabolic distance between points? Check that for $u : P_1 \to \mathbb{R}$, $u^\lambda : P_{\lambda^{-1}} \to \mathbb{R}$. If u satisfies the heat equation, what does u^λ satisfy? What about another linear equation?

Exercise 2. Suppose that $k, l \in \mathbb{N}$, $\alpha, \beta \in (0,1)$ are such that $l + \beta < k + \alpha$, and suppose that a sequence $u_i : \overline{\Omega_T} \to \mathbb{R}$ has $|u_i|_{k+\alpha} < C$. Show that there is a subsequence u_{i_j} which converges to some u in $C^{l+\beta;\frac{l+\beta}{2}}$. Furthermore, show that $u \in C^{k+\alpha;\frac{k+\alpha}{2}}(\overline{\Omega_T})$.

Exercise 3. Suppose that $u \in C^{k+\alpha;\frac{k+\alpha}{2}}(\Omega_T)$ where Ω_T is a set of compact closure. Show that there exists a unique continuous extension of u to $\overline{\Omega_T}$, such that all derivatives are continuous up to the boundary.

Exercise 4. Prove that $C^{k+\alpha;\frac{k+\alpha}{2}}(\Omega_T)$ is a Banach space.

Exercise 5. Look up the extension theorem for elliptic Hölder spaces in Gilbarg and Trudinger [3, Section 6.9 on p136]. Extend this theorem to parabolic spaces, that is, prove the following:

Suppose that $u \in C^{k+\alpha;\frac{k+\alpha}{2}}(\Omega_T)$ where Ω_T is a set of compact closure and $\partial\Omega$ is smooth (in fact, it is enough for it to be written locally as a $C^{2,\alpha}$ graph). Suppose additionally that $\overline{\Omega} \subset \Omega' \subset \mathbb{R}^{n+1}$ for some open Ω'. Then there exists a $C = C(\Omega, \Omega') > 0$ such that any $u \in C^{2+\alpha;\frac{2+\alpha}{2}}(\Omega_T)$ may be extended to $\tilde{u} : \Omega' \times (-1, T+1)$ where $\tilde{u} = 0$ in a neighbourhood of $\partial(\Omega' \times (-1, T+1))$ and

$$|\tilde{u}|_{k+\alpha, \Omega' \times (-1,T+1)} \leq C |u|_{k+\alpha, \Omega_T}$$

Exercise 6. Consider the two norms

$$|u|_{1+\alpha} = |u|_0 + [Du]_\alpha + \langle u \rangle_{\frac{1+\alpha}{2}}$$
$$|u|'_{1+\alpha} = |u|_0 + [Du]_\alpha \ .$$

By considering functions $f(x,t) = f(t)$ or otherwise, show that these norms are not equivalent. Is $C^{\alpha;\frac{\alpha}{2}}$ contained in the function spaces defined by these norms?

Exercise 7. Complete the proof of Lemma 6. Similarly complete the proof of Corollary 9. Prove Corollary 11 in full.

Exercise 8. Prove that for any $0 < \beta < \alpha < 1$, if $u \in C^{\alpha;\frac{\alpha}{2}}(\overline{\Omega}_T)$ then $[u]_{\beta,\Omega} \le 2[u]_{\alpha,\Omega}^{\frac{\beta}{\alpha}} \operatorname{osc}(u)^{1-\frac{\beta}{\alpha}}$. Here $\operatorname{osc}(u) = \sup u - \inf u$.

Exercise 9. Suppose that $f, g \in C^{\alpha;\frac{\alpha}{2}}(\Omega_T)$. Show that

$$[f \cdot g]_\alpha \le |f|_0 [g]_\alpha + |g|_0 [f]_\alpha \ .$$

Exercise 10. Suppose that $f \in C^{\alpha;\frac{\alpha}{2}}(\Omega_T)$ and $\Phi : \mathbb{R} \to \mathbb{R}$ is Lipschitz then

$$[\Phi(f)]_\alpha \le |\Phi|_{\mathrm{Lip}} [f]_\alpha \ .$$

More generally, if $\Phi : \mathbb{R} \to \mathbb{R}$ is C^β show that

$$[\Phi(f)]_{\beta\alpha} \le C[\Phi]_\beta [f]_\alpha \ .$$

Exercise 11. Suppose that $[f]_\beta < C$ for all $\beta < \alpha$. Show that $[f]_\alpha \le C$.

2. Short time existence and bootstrapping

2.1. **Schauder theory.** We now prove a short-time existence theorem on the sphere. We will need the following, which will for now be taken as a fact.

Theorem 13 (Local Schauder estimates)**.** *Suppose that $\Omega, \Omega' \subset \mathbb{R}^n$ are open sets of compact closure such that $\overline{\Omega} \subset \Omega'$. We define*

$$Lu := u_t - a^{ij}(x,t) D^2_{ij} u - b^i(x,t) D_i u - c(x,t) u \ ,$$

$a^{ij}, b^i, c, f \in C^{k+\alpha;\frac{k+\alpha}{2}}(\Omega_T)$ *and suppose that* $u_0 \in C^{k+2,\alpha}(\Omega)$*. Suppose that* $u \in C^{k+2+\alpha;\frac{k+2+\alpha}{2}}(\Omega'_T)$ *satisfies*

$$\begin{cases} Lu = f & \text{for } (x,t) \in M_T \\ u(x,0) = u_0(\cdot) & \text{for } x \in M \end{cases} .$$

Then there exists a constant $C > 0$, depending on n the coefficients of L, Ω and Ω' such that

$$|u|_{k+2+\alpha,\Omega_T} \le C(|u_0|^0_{k+2+\alpha,\Omega'} + |f|_{\alpha,\Omega'_T} + |u|_{0,\Omega'_T}) \ .$$

For a proof of this, see Ladyženskaja, Solonikov and Uralt'seva, [6, Chapter 5, Theorem 10.1 pages 351-352]

Theorem 14 (Schauder theory on a manifold)**.** *Suppose that on M^n we have the linear parabolic PDE*

$$Lu := u_t - a^{ij}(x,t) D^2_{ij} u - b^i(x,t) D_i u - c(x,t) u$$

where indices are covariant derivatives wrt to a background metric g_0. Suppose that $a^{ij}, b^i, c, f \in C^{k+\alpha;\frac{k+\alpha}{2}}(M_T)$ *and that* $u_0 \in C^{k+2+\alpha}(M)$*. Then, there exists a unique solution* $u \in C^{k+2+\alpha;\frac{k+2+\alpha}{2}}(M_T)$ *to the equation*

$$\begin{cases} Lu = f & \text{for } (x,t) \in M_T \\ u(x,0) = u_0(\cdot) & \text{for } x \in M \end{cases} ,$$

and furthermore this solution satisfies

$$|u|_{C^{k+2+\alpha;\frac{k+2+\alpha}{2}}(M_T)} \le C(|f|_{C^{k+\alpha;\frac{k+\alpha}{2}}(M_T)} + |u_0|^0_{C^{k+\alpha}(M)} + |u|_{C^0(M_T)}) \ .$$

Remark 1. It will sometimes be useful to have the above estimate but without the $|u|_{C^0(M_T)})$ term. We may do this, but at the expense of allowing the constant C to depend on T. By maximum principle we may see that $|u(\cdot,t)| \le (C(1+t)|u_0|_{2,\Omega} + |f|t)e^{|c|_{0,M_T}t}$:

For example, writing

$$L^0 u := u_t - a^{ij}(x,t)D^2_{ij}u - b^i(x,t)D_i u$$

then extending u_0 to be constant in time we have that

$$L^0(u-u_0) = f + cu - L^0 u_0 \le |c|_{0,M_T}(u-u_0) + |f|_0 + C|u_0|_2$$

and so

$$L^0[(u-u_0)e^{-|c|_{0,M_T}t} - t(|f|_0 + C|u_0|_2)] \le 0$$

and so, applying the maximum principle,

$$u(\cdot,t) - u_0(\cdot) \le (C|u_0|_{2,\Omega} + |f|)te^{|c|_{0,M_T}t} \ .$$

A similar estimate gives a lower bound. Using this we may alternatively estimate

$$|u|_{C^{k+2+\alpha;\frac{k+2+\alpha}{2}}(M_T)} \le C(T)(|f|_{C^{k+\alpha;\frac{k+\alpha}{2}}(M_T)} + |u|_{C^0(M_T)}) \ .$$

Some proof ideas. Our first aim is to show that for any solution $u \in C^{k+2+\alpha;\frac{k+2+\alpha}{2}}$, the above estimate holds.

To do this, firstly note that on any of the charts u satisfies a linear equation on each chart as in Theorem 13. Next, by choosing domains $\Omega = B_{1-\delta} \subset \Omega' = B_1(0)$ we observe that for δ sufficiently small, every point on M will be the pre image of at least one of the $B_{1-\delta}$'s. From Theorem 13 we get Schauder estimates on $B_{1-\delta}$. On the other hand, on the "overlap" regions $B_1(0) \setminus B_{1-\delta}(0)$, we may use interior estimates from other charts along with the coordinate transformations (which are smooth) to see that on $B_{1,T}(0)$

$$|u \circ \phi_i^{-1}|_{C^{k+2+\alpha;\frac{k+2+\alpha}{2}}(B_{1,T}(0))} \le C(|f|_{C^{k+\alpha;\frac{k+\alpha}{2}}(M_T)} + |u_0|^0_{C^{k+\alpha}(M)} + |u|_{C^0(M_T)}) \ .$$

Summing over all i now gives the claimed estimate.

Existence now follows from the method of continuity (Theorem 34). Without loss of generality (by changing f), we may assume that $u_0 \equiv 0$. We define L_1 to be the operator of interest, and we take L_0 to be the heat equation – this operator is surjective (this can be proven using either explicit solutions using the spectrum of the Laplacian or using weak solutions directly). Then the estimate above implies that we may apply the method of continuity to get

Uniqueness of the solution now follows from the maximum principle. □

2.2. **Short time existence using Newton iteration.** We want to prove the following:

Theorem 15. *We consider the fully nonlinear PDE*

$$P(u) = u_t - F(D^2u, Du, u, x, t)$$

Suppose that $u_0 \in C^{2+\alpha}(\Omega)$ is λ-Λ admissable for some $0 < \lambda < \Lambda$. Then for any $0 < \beta < \alpha$, there exists $\tau = \tau(|u_0|_{2+\alpha}, |F|^0_3, \beta) > 0$ such that there is an unique solution $u \in C^{2+\beta;\frac{2+\beta}{2}}(M_\tau)$ to

$$\begin{cases} P(u) = 0 & \text{on } \Omega_\tau \\ u(\cdot,0) = u_0(\cdot) & \text{on } \Omega \end{cases} \ . \tag{14}$$

Remark 2. Note that when proving Theorem 15, we may assume that $u_0 = 0$. If it is not, then (extending u_0 to be constant in time) we may simply define $\tilde{u} = u - u_0$ and we see that $\tilde{u}$ satisfies

$$\begin{cases} \tilde{P}(\tilde{u}) = 0 \\ \tilde{u}(\cdot, 0) = 0 \end{cases}$$

where $\tilde{F}(D^2\tilde{u}, D\tilde{u}, \tilde{u}, x, t) = F(D^2\tilde{u} + D^2u, D\tilde{u} + Du, \tilde{u} + u, x, t)$. We may check (exercise) that $\tilde{F}$ is parabolic at $\tilde{u}$ iff F is parabolic at u. As a result, from now on, we will assume that our initial data is zero.

We recall "Newton Iteration" – a numerical way of finding the zero of a function. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Then (supposing that it exists) we aim to find a zero of f iteratively: Firstly, pick any x_1. Given x_i , let x_{i+1} be the point at which the linearisation of f at x_i meets the x-axis (in this case, $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ and this will clearly only work if $f'(0) \neq 0$).

INSERT PICTURE!

Then hopefully under sufficient assumptions this sequence will converge to some $x \in \mathbb{R}$ with $f(x) = 0$ as in the picture. To get rigorous results, we could show that (under suitable assumptions) in a neighbourhood of $f(0)$, the map $x \mapsto x - \frac{f(x)}{f'(x)}$ is a contraction (exercise - find the assumptions and compare them to what we need below...).

To prove the short time existence for a nonlinear Parabolic PDE, we now use the same idea to find a solution to $P(u) = 0$. For us, the x axis is replaced with a subset of $C^{2+\alpha;\frac{2+\alpha}{2}}(\Omega_T)$, the y-axis would be replaced with $C^{\alpha;\frac{\alpha}{2}}(\Omega_T)$, and f is replaced with $P(u)$. More specifically, we will consider the mapping Φ where $\Phi(u) = v$ where v is the solution to

$$\begin{cases} L_u v = L_u u - P(u) & \text{on } \Omega_\tau \\ v(\cdot, 0) = 0 & \text{on } \Omega \end{cases}, \tag{15}$$

and aim to show that Φ is a contraction. That is, "$\Phi(u) = u - L_u^{-1}(P(u))$", compare this formula to the Newton iteration formula.

Proof of theorem 15. Our aim is to find a $\tau > 0$ and a suitable closed subset of $C^{2+\alpha;\frac{2+\alpha}{2}}(M_\tau)$ such that on this set, Φ is a contraction mapping on that set. Then contraction mapping theorem (Theorem 35) gives a fixed point. Note that, at fixed point of Φ, $\Phi(u) = u$, so $L_u u = L_u u - P(u)$, that is, $P(u) = 0$ as required.

We start with $\tau = 1$ and we will make τ smaller repeatedly through our proof. Our first simplification allows us to essentially remove the initial data from the problem:

Step 1: Simplifying initial data by defining v_0. We first restrict time so that for $t \in [0, \tau]$, L_0 is uniformly parabolic[1]. We consider a solution to the equation

$$\begin{cases} L_0 v_0 = F(0, 0, 0, x, t) & \text{on } \Omega_\tau \\ v_0(\cdot, 0) = 0 & \text{on } \Omega \end{cases}. \tag{16}$$

[1]include def

Note that there exists a solution to (16) $v_0 \in C^{2+\alpha;\frac{2+\alpha}{2}}$ by Theorem **??**. Furthermore, $|v_{0,t}| < C$ and so $|v_0| \leq C\tau$. Therefore by interpolation, for any $0 < \beta < \alpha$ we know that by restricting τ, $|v_0|_{2+\beta} \leq C\tau^{\frac{\alpha-\beta}{2+\alpha}}$ and so v_0 may be made arbitrarily small in $C^{2+\beta;\frac{2+\beta}{2}}$.

Step 2: Define the domain of Φ. We define the closed, bounded, convex set

$$D_\tau^{R,v_0} := \left\{ u \in C^{2+\beta;\frac{2+\beta}{2}}(M_\tau) : |u - v_0|_{2+\beta,\Omega_\tau} \leq R,\ u(\cdot,0) = 0 \right\} .$$

As $\Gamma_{\lambda,\Lambda}$ is open, and L_{u_0} is strictly parabolic, there exists an $R_0 > 0$ such that for any $u \in D_\tau^{R,v_0}$ L_u is parabolic. Our claim is that there is an $0 < R < R_0$ and a $\tau > 0$ such that $\Phi : D_\tau^{R,v_0} \to D_\tau^{R,v_0}$ is well defined and is a contraction. There are two parts to this - showing that this is well defined (i.e. $\mathrm{Im}(\Phi) \subset D_\tau^{R,u_0}$) and showing that this is indeed a contraction. In particular, note that by restricting R and τ, we may estimate that for any $u \in D_\tau^{R,v_0}$, $|u|_{2+\beta}$ may be made arbitrarily small (due to our estimates on v_0). For brevity below, we will only show estimates for the highest order terms (which are the most important ones) and leave completion of this proof as an exercise. However, we prove enough to give the theorem for $F = F(D^2u)$.

Step 3: For R and τ sufficiently small, Φ is a contraction. The idea here is pretty much the same as the previous step. Suppose that $u_1, u_2 \in D_\tau^{R,u_0}$ with $v_1 = \Phi(u_1)$, $v_2 = \Phi(u_2)$. Then

$$\begin{cases} L_{u_1}(v_1 - v_2) = f & \text{on } \Omega_\tau \\ (v_1 - v_2)(\cdot, 0) = 0 & \text{on } \Omega \end{cases} , \tag{17}$$

where

$$\begin{aligned} f &= L_{u_1}u_1 - P(u_1) - L_{u_1}v_2 \\ &= L_{u_1}u_1 - P(u_1) - L_{u_2}v_2 + L_{u_2}v_2 - L_{u_1}v_2 \\ &= L_{u_1}u_1 - L_{u_2}u_2 + P(u_2) - P(u_1) + L_{u_2}v_2 - L_{u_1}v_2 \ . \end{aligned}$$

As above, our plan will be to show that, for any $\epsilon > 0$ we may find $\tau > 0$ and $R > 0$ small enough so that $|f|_\alpha \leq \epsilon |u_1 - u_2|_{2+\alpha}$. Then using Schauder estimates and taking ϵ to be small enough will show that $|\Phi(u_1) - \Phi(u_2)|_{2+\alpha} \leq \frac{1}{2}|u_1 - u_2|_{2+\alpha}$.

Writing $u_s = su_2 + (1-s)u_1$ and $a(s) = F(u_s)$ then using Taylor's theorem of the form $a(1) = a(0) + a'(0) + \int_0^1 (1-s)a''(s)ds$,

$$P(u_2) - P(u_1) = L_{u_1}(u_2 - u_1) + \int_0^1 (1-s)a'' ds$$

so

$$f = (L_{u_1}u_2 - L_{u_2}u_2) - (L_{u_1}v_2 - L_{u_2}v_2) + \int_0^1 (1-s)a'' ds \ .$$

For brevity we now restrict to $F = F(D^2u)$: Writing $u_s = su_1 + (1-s)u_2$ and then using $b(s) = L_{u_s}u_2$ as above we have

$$L_{u_1}u_1 - L_{u_2}u_2 = \int_0^1 b'(s)ds = \int_0^1 F^{ij,kl}|_{u_s} ds D_{ij}u_2 D_{kl}(u_1 - u_2) + \text{plenty of other terms.}$$

We have that $|\int_0^1 F^{ij,kl}|_{u_s} ds|_\beta < C|D^3F|(|D^2u_1|_\beta + |D^2u_2|_\beta)$ (exercise), using the following multiplicative identity for Hölder seminorms, (exercise)

$$[ef]_\beta \le |e|_0[f]_\beta + |f|_0[e]_\beta \ ,$$

we have that

$$|L_{u_1}u_2 - L_{u_2}u_2|_\beta \le C(|D^2u_2|_0|u_1-u_2|_{2+\beta} + |D^2u_2|_\beta|u_1-u_2|_2) \le C|D^2u_2|_\beta|u_1-u_2|_{2+\beta} \ .$$

In an identical manner we have that

$$|L_{u_1}v_0 - L_{u_2}v_0|_\beta \le C|D^2v_0|_\beta|u_1-u_2|_{2+\beta} \ .$$

Finally, we have that

$$\int_0^1 (1-s)a''ds = \int_0^1 (1-s)F^{ij,kl}|_{u_s} ds D^2_{ij}(u_2-u_1)D^2_{kl}(u_2-u_1)$$

so by a similar estimate on the integral term

$$\left|\int_0^1 (1-s)a''ds\right|_\alpha \le C|u_1-u_2|_2|u_1-u_2|_{2+\beta} \ .$$

Estimating as in Step 1, we see that $u_1, u_2 \in D^{R,v_0}_\tau$,

$$|f|_\beta \le C(\tau^{\frac{\alpha-\beta}{2+\alpha}} + R)|u_1-u_2|_{2+\beta}$$

as required.

Step 4: Given any $R < R_0$, there exists a τ such that Φ maps as claimed, that is $\Phi : D^{R,u_0}_\tau \to D^{R,u_0}_\tau$. In the definition of Φ, we aim to control $w = v - v_0$ in $C^{2+\beta;\frac{2+\beta}{2}}$. We see that rewriting (15), w satisfies

$$\begin{cases} L_u(v-v_0) = f & \text{on } \Omega_\tau \\ w(\cdot,0) = 0 & \text{on } \Omega \end{cases} , \tag{18}$$

where $f := L_u u - P(u) - L_u v_0$.

We claim that there is a $p = p(\beta) > 0$ so that we may bound by $|f|_\beta < C(R, v_0)\tau^p$ where the constant is allowed to depend on R. By making τ smaller, we may make $|f|_\alpha$ arbitrarily small. Therefore, equation (18) and Schauder estimates (Theorem **??**) now imply that we can make $|v - v_0|_{2+\beta}$ arbitrarily small, and in particular, less than R, and so we are done modulo showing this claim which we now sketch a proof of.

We compute

$$\begin{aligned} f &:= L_u u - P(u) - L_u v_0 \\ &= -P(u) - L_0 v_0 + L_u u + (L_0 v_0 - L_u v_0) \\ &= P(0) - P(u) + L_u u + (L_0 v_0 - L_u v_0) \\ &= F|_u - F|_0 - F^{ij}|_u D^2_{ij}u - F^{p_i}|_u D_i u - F_z|_u u + L_0 v_0 - L_u v_0 \end{aligned}$$

where we are using the notation $F|_u = F(D^2u, Du, u, x, t)$, $F^{ij}|_u = F^{ij}(D^2u, Du, u, x, t)$, and so on.

Writing $u_s = su$ we define $c(s) = P|_{u_s}$, again using Taylor's theorem we see that $P|_u = P|_0 + L_0 u + \int_0^1 (1-s)c''(s)ds$. Therefore this time,

$$f = -\int_0^1 (1-s)c''ds + (L_u u - L_0 u) + (L_0 v_0 - L_u v_0) \ .$$

For brevity we now restrict to $F = F(D^2u)$: Similarly to the integral term in the previous section, we may estimate the integral term by

$$\left| -\int_0^1 (1-s)c''ds \right|_\beta \leq C|u|_2 |u|_{2+\beta} \leq C(R)\tau^{\frac{\beta}{2+\beta}}$$

where we used that $|u(\cdot,t)| \leq |u|_{2+\beta} t$ so by interpolation, $|u|_2 \leq \tau^{\frac{\beta}{2+\beta}}$. Similarly, using an identical estimate to the previous step we get that

$$|L_u u - L_0 u|_\beta \leq C|u|_2 |u|_{2+\beta} \leq C(R)\tau^{\frac{\beta}{2+\beta}}$$
$$|L_u v_0 - L_0 v_0|_\beta \leq C(|v_0|_2 |u|_{2+\beta} + |u|_2 |v_0|_{2+\beta}) \leq C(R)\tau^{\frac{\beta}{2+\beta}}$$

and so we have the claim.

□

2.3. **Short time existence using the inverse function theorem.** Danger: This section of the notes is uncorrected and is essentially in Gerhardt's book - have a look at this there!

This method of proof is quite nicely written out in Gerhardt's book [2, Theorem 2.5.7 on page 106]. However, I also comment that a lot of the work is hidden in the statement "Φ is continuously Fréchet differentiable" (see [8, Lemma 1.8, p9-12] for some details on this).

Recall that a mapping between Banach spaces $\Phi : V \to W$ is Frechet differentiable if there exists a linear mapping $D\Phi : V \to W$ such that

$$\|\Phi(x+h) - \Phi(h) - D\Phi(h)\|_W \leq o(\|h\|_V)$$

where $\|h\|^{-1}o(\|h\|) \to 0$ as $\|h\| \to 0$. On an open set $U \subset V$ this is said to be continuously differentiable if at every point $u \in U$ $D\Phi|_u$ exists and is continuous as a mapping $D\Phi : U \to L(V,W)$. We have the following

Theorem 16 (Banach space Inverse Function Theorem)**.** *Suppose that V and W are Banach spaces and that for some open $U_V \subset V$, $\Phi : U \to W$ is continuously differentiable mapping. Furthermore suppose that at $u \in U$, $D\Phi|_u : V \to W$ is a bounded linear isomorphism. Then there exists a neighbourhood of $\Phi(u)$, $U_W \subset W$ and a continuously differentiable map $\Psi : U_W \to V$ such that $\Phi(\Psi(f)) = f$ for all $f \in U_W$. Moreover $\Psi(y)$ is unique for sufficiently small if we restrict to a sufficiently small neighbourhood U_V.*

Theorem 17 (Short time existence via IFT)**.** *Suppose that $F(x,t,z,p,r)$ is smooth (in C^3 in all of its entries is enough) and, for some $\Lambda, \epsilon > 0$ let $\Gamma \subset C^2(\Omega)$ be an open set such that for all $u \in \Gamma$ and $(x,t) \in \Omega_\epsilon$*

$$\Lambda^{-1}|\xi|^2 \leq \frac{\partial F}{\partial r_{ij}}\Big|_{(x,t,u,Du,D^2u)} \xi_i \xi_j \leq \Lambda |\xi|^2 \ .$$

Let $u_0 \in \Gamma \cap C^{2+\alpha}(\Omega)$. Then for any $0 < \beta < \alpha$ there exists a $\tau = \tau(\beta, u_0, \epsilon) > 0$ such that the PDE

$$\begin{cases} u_t = F(x,t,u,Du,D^2u) \ , \\ u(\cdot,0) = u_0(\cdot) \end{cases} \tag{19}$$

has a unique solution $u \in C^{2+\beta;\frac{2+\beta}{2}}(\Omega_\tau)$.

Proof. This proof is in several steps. To begin, we need to start by extending our initial data in a sensible way to simplify computations later.

Step 1: We define $\tilde{u}$ to be the solution of the equation

$$\begin{cases} \tilde{u}_t - \Delta\tilde{u} = F(x,t,u_0,Du_0,D^2u_0) - \Delta u_0 \\ \tilde{u}(\cdot,0) = u_0(\cdot) \end{cases}$$

where Δ is the standard Laplacian. By Theorem 14 such a solution exists with $\tilde{u} \in C^{2+\alpha;\frac{2+\alpha}{2}}(\Omega_T)$. As Γ is open, this means that there is a $\tau_0 > 0$ so that $\tilde{u}(\cdot,t) \in \Gamma$ for all $0 \leq t \leq \tau_0$.

Step 2: We define $f \in C^{\alpha;\frac{\alpha}{2}}(\Omega_{\tau_0})$ by

$$\tilde{f} = \tilde{u}_t - F(x,t,\tilde{u},D\tilde{u},D^2\tilde{u}) \ .$$

In particular, by our choice of $\tilde{u}$, we have that $\tilde{f}(\cdot,0) = 0$.

Step 3: We consider the nonlinear operator $\Phi : C^{2+\beta;\frac{2+\beta}{2}} \to C^{\beta;\frac{\beta}{2}} \times C^{2+\beta}$ which is given by

$$\Phi(u) = (u_t - F(x,t,u,Du,D^2u), u(0))$$

and is defined on a neighbourhood of $\tilde{u} \in V \subset C^{2+\beta;\frac{2+\beta}{2}}(\Omega_{\tau_0})$ and has image in $W = C^{\beta;\frac{\beta}{2}}(\Omega_{\tau_0}) \times C^{2+\beta}(\Omega)$. It turns out that, as F is in $C^{2+\alpha}$ in its entries, Φ is continuously (Frechet) differentiable (longish exercise) and its derivative $D\Phi$ evaluated at $\tilde{u}$ is the operator is the operator L where

$$\mathcal{L}v = (L_{P,\tilde{u}}v, v(0))$$

and $P_{\tilde{u}}$ is the linearisation of the PDE at $\tilde{u}$ as in (2). In particular $\pi_1 \circ \mathcal{L}$ is a parabolic linear operator with coefficients in $C^{\beta;\frac{\beta}{2}}(\Omega_{\tau_0})$ and so by Theorem 14 this is a linear isomorphism from V to W.

Applying the Banach space Inverse Function Theorem, Φ restricted to $B_\rho(\tilde{u}) \subset V$ is a C^1 diffeomorphism onto an open neighbourhood of $(\tilde{f}, u_0) \in U \subset W$.

Step 4: Let $\epsilon > 0$ be small and choose $\chi_\epsilon \in C^\infty([0,1])$ so that $0 \leq \chi_\epsilon \leq 1$ and $0 \leq \chi(\epsilon \leq 2\epsilon^{-1}$ and

$$\chi_\epsilon(t) = \begin{cases} 0 & 0 \leq t \leq \epsilon \\ 1 & 2\epsilon \leq t \leq 1 \end{cases}$$

and define $f_\epsilon := \tilde{f}\chi_\epsilon(t)$. In Lemma 18 below, we will see that there is a constant C such that $|f_\epsilon|_{C^{\alpha;\frac{\alpha}{2}}(Q_{\tau_0})} < C$. Then, we have that $|f_\epsilon - \tilde{f}|_{C^0(Q_{\tau_0})} \to 0$ (by continuity of $\tilde{f}$ and the fact that $\tilde{f} = 0$ at $t = 0$) and so by either Ehrling's lemma, interpolation or Arzèla–Ascoli, for any $0 < \beta < \alpha$,

$$\lim_{\epsilon \to 0} |f_\epsilon - \tilde{f}|_{C^{\beta;\frac{\beta}{2}}(Q_{\tau_0})} = 0 \ .$$

As a result, for ϵ small enough $(f_\epsilon, u_0) \in U$, and so there exists a unique solution $u \in B_\rho(\tilde{u})$ to $\Phi(u) = (f_e, u_0)$. In particular, this implies that (19) has a solution for $0 \leq t \leq \epsilon$.
Step 5: Global uniqueness - this follows from the maximum principle.

□

Lemma 18. *Suppose that* $\tilde{f} \in C^{\alpha;\frac{\alpha}{2}}(\Omega_T)$ *and* $\tilde{f}(\cdot, 0) = 0$. *Then for* f_ϵ *as defined in Step 4 of the proof of Theorem 15, we have that there exists a constant* C *such that* $|f_\epsilon|_{C^{\alpha;\frac{\alpha}{2}}(\Omega_T)} < C$.

Proof. As χ_ϵ is constant in space, we only need consider the semi-norm with respect to t. Suppose that $|\tilde{f}|_{C^{\alpha;\frac{\alpha}{2}}(\Omega_T)} = C_f$.

We temporarily pick a point $x \in \Omega$ and abuse notation by writing $f(t) = f(x,t)$ Suppose wlog that $0 \leq t_1 < t_2 < \tau_T$ and note that

$$|f_\epsilon(t_1) - f_\epsilon(t_2)| \leq |\tilde{f}(t_1) - \tilde{f}(t_2)||\chi_\epsilon(t_2)| + |\tilde{f}(t_1)||\chi_\epsilon(t_1) - \chi_\epsilon(t_2)| \ .$$

The first term is bounded by a multiple of $|t_2 - t_1|^{\frac{\alpha}{2}}$ by our assumption on $\tilde{f}$. We now estimate the second term by considering several cases:

Suppose first that $t_1 \leq 2\epsilon$ and $t_2 \leq 3\epsilon$. In this case

$$|\tilde{f}(t_1)||\chi_\epsilon(t_1) - \chi_\epsilon(t_2)| \leq 2C_f t_1^{\frac{\alpha}{2}} \epsilon^{-1}|t_1 - t_2| \leq 2C_f (2\epsilon)^{\frac{\alpha}{2}} \epsilon^{-1}|t_1 - t_2| \leq C|t_1 - t_2|^{\frac{\alpha}{2}}$$

where we used that in this case $|t_1 - t_2| \leq 3\epsilon$.

Now suppose that $t_1 \leq 2\epsilon$ and $t_2 > 3\epsilon$ so that $\epsilon \leq t_2 - t_1$. In this case

$$|\tilde{f}(t_1)||\chi_\epsilon(t_1) - \chi_\epsilon(t_2)| \leq C_f t_1^{\frac{\alpha}{2}} \leq C_f 2^{\frac{\alpha}{2}} \epsilon^{\frac{\alpha}{2}} \leq C|t_2 - t_1|^{\frac{\alpha}{2}} \ .$$

Finally if $t_1 > 2\epsilon$ the second term is zero and so we are done. □

2.4. **Bootstrapping.** We now show that if F has greater regularity then we may conclude that so does our PDE solution. A good reference for this is [2, Theorem 2.5.10, p111]

Theorem 19. *Suppose that for some* $l \in \mathbb{N}$, $F(r,p,z,x,t)$ *is smooth (bounded in* $C^{2+l+\alpha}$ *in* z,p,r *and in* $C^{2+l+\alpha;\frac{2+l+\alpha}{2}}(\Omega_T)$ *in* (x,t)*) and that* $u_0 \in C^{2+l+\alpha}(\Omega)$. *Then given a solution* $u \in C^{2+\beta;\frac{2+\beta}{2}}(\Omega_T)$ *of (19) which is uniformly parabolic with* $u(\cdot,t) \in \Gamma_{\lambda,\Lambda}$ *for all* $t \in [0,T)$, *then in fact this solution satisfies* $u \in C^{2+l+\beta;\frac{2+l+\beta}{2}}(\Omega_T)$ *with uniform estimates on* u, *that is for all* $m \leq l$ *there exists a* $C_m = C_m(|u|_{2+\beta}, |u_0|^0_{2+m+\alpha}, F, \alpha, \beta)$ *such that*

$$|u|_{2+m+\beta} < C_m \ .$$

Furthermore, if $l \geq 2$ *then the above holds with* $\beta = \alpha$.

Proof. By interpolation, it is enough to bound $[D^{k,\frac{k}{2}}u]_\beta$. We go about this in two steps.

Step 1: Space derivatives. We prove the required bounds on a small ball $B_\rho(0) \subset B_{3\rho}(0)$ contained in a single coordinate chart of S^n.

Using the standard basis $e_1, \dots, e_n$, for $e \in \{e_1, \dots, e_n\}$ define

$$\Delta_h f := \frac{f(x+he) - f(x)}{h} \ .$$

We consider $v = \Delta_h u$ with initial data $v_0 = \Delta_h u_0$. Writing

$$u_\tau(x,t) = \tau u(x+he,t) + (1-\tau)u(x,t)$$

we see that v satisfies

$$\begin{aligned} v_t &= h^{-1}(F(D^2u(x+he), Du(x+he)\ldots) - F(D^2u(x), Du(x), \ldots)) \\ &= h^{-1}\int_0^1 \frac{d}{d\tau}(F(D^2u_\tau, Du_\tau, u_\tau, x+\tau he, t)d\tau \\ &= a^{ij}(h)D^2_{ij}v + b^i(h)D_iv + c(h)v + f(h) \end{aligned}$$

where

$$\begin{aligned} a^{ij}(h) &= \int_0^1 F^{ij}|_{u_\tau}d\tau, \qquad b^i(h) = \int_0^1 F_{p^i}|_{u_\tau}d\tau \\ c(h) &= \int_0^1 F_z|_{u_\tau}d\tau, \qquad f(h) = \int_0^1 F_e|_{u_\tau}d\tau \ . \end{aligned}$$

As, $\Gamma_{\lambda,\Lambda}$ is convex and u is strictly away from the boundary of $\Gamma_{\frac{\lambda}{2},2\Lambda}$, u_τ is admissable and so the above PDE is parabolic. Furthermore, all coefficients are in $C^{\beta;\frac{\beta}{2}}(\Omega_T)$ with norms independent of h.

Now let $\chi \in C^\infty_c(B_{2\rho}(0))$ be a cutoff function which is 1 on $B_\rho(0)$. We set $w = \chi v$ and we see that w satisfies

$$\begin{cases} w_t - a^{ij}D^2_{ij}w - \tilde{b}^iD_iw - \tilde{c}w = f \\ w(\cdot, 0) = v_0\chi \end{cases}$$

where again, this is uniformly parabolic with coefficients in $C^{\beta;\frac{\beta}{2}}$. Applying Theorem 14 we have that $|w|_{C^{2+\alpha;\frac{2+\alpha}{2}}(\Omega_T)} < C$ where the constant is independent of h. Taking a limit as $h \to 0$, this in particular implies that D_ku exists everywhere, is continuous and in fact $|D_eu|_{2+\beta,B_{\rho,T}} < C$. We repeat this for all e to get $Du \in C^{2+\beta;\frac{2+\beta}{2}}((B_\rho)_T)$. We are now able to differentiate the equation to get

$$\begin{cases} D_ku_t &= F_{ij}(D_ku)_{ij} + F_{p^i}(D_ku)_i + F_zD_ku + F_e \\ D_ku(\cdot,0) &= D_ku_0(\cdot) \end{cases} \tag{20}$$

From this, we immediately see that $|D_ku_t|_{\beta,B_{\delta,T}} < C$. Furthermore , we may now repeat the above difference equation process, by induction we have that for any multi-index γ with $|\gamma| \leq l$,

$$|D_\gamma u|_{2+\beta,B_{\rho,T}} < C, \qquad |D_tD_\gamma u|_{\beta,B_{\rho,T}} < C \ . \tag{21}$$

By repeating the above on balls covering M we get the same estimate but on M_T.

Step 2: Time derivatives for $l = 2$. Suppose that $l = 2$ (if $l = 1$ we are already done). We already know that $D^2u_t \in C^{\beta;\frac{\beta}{2}}(\Omega_T)$ and $u_t(\cdot,0) \in C^{2+\alpha}(B_\rho(0))$ and $u_t(\cdot,t) \in C^{2+\beta}(\Omega)$. We may repeat the difference quotient method (in the time direction) and use a cutoff in time as well as space. This yields that for any δ, $u \in C^{4+\beta;\frac{4+\beta}{2}}(B_\rho \times [\delta, T-\delta])$ where the estimate on the norm depends on δ and explodes as $\delta \to 0$. However, this does mean that we may now differentiate (19) in time to see that $v := u_t$ satisfies the PDE

$$v_t = F^{ij}D^2_{ij}v + F_{p^i}D_iv + F_zv + F_t \ .$$

and, from our space estimates, we have that the coefficients of this equation are uniformly in $C^{2+\beta;\frac{2+\beta}{2}}(B_\rho \times [0,T])$ (with estimates which are independent of δ) and $v(\cdot,\delta)$ is uniformly bounded in $C^{2+\beta}(\Omega)$ independently of δ by (21). Applying Theorem 14 we have that

$|v|_{C^{2+\beta;\frac{2+\beta}{2}}(B_\rho\times[\delta,T])} < C$ for all δ with the constant independent of δ. Sending $\delta \to 0$ we get estimates on $B_{\rho,T}$. Therefore $u \in C^{4+\beta;\frac{4+\beta}{2}}(B_{\rho,T})$.

Step 3: Bumping up β to α for $l = 2$. Regularity can now be improved - we have higher order estimates on u, therefore the coefficients in equation (20) are in $C^{\alpha;\frac{\alpha}{2}}$. We may therefore repeat the above estimates, but with β replaced with α.

Step 4: $l > 2$. This case now follows by induction - we repeat Step 2 $\lfloor\frac{l}{2}\rfloor$ times, completing the proof. □

3. Finding the missing α using Krylov–Safonov

We now suppose that we are in the situation in which we have somehow proved that, for as long as our solution exists, it is uniformly parabolic and has $|u|_2 < C$ for some fixed C . Usually, this takes some work and several applications of the maximum principle, and it is not always possible! However, while our bootstrapping estimates require $|u|_{2+\alpha} < C$ to get uniform smooth estimates i.e. for all l there exists a C_l such that

$$|u|_{l+\alpha} < C_l \ .$$

Indeed, at the end of our existence time interval there is currently nothing stopping $|D^3u(\cdot,t)|_0^0 \to \infty$ as $t \to T$. We now get *uniform* $C^{2+\alpha;\frac{2+\alpha}{2}}$ estimates from $C^{2,1}$ estimates which gets around this putative issue.

Specifically, we demonstrate that under reasonable conditions on F, $|u|_{C^{2,1}(M_T)} < C$ for some fixed C. However, while our bootstrapping estimates imply that $|u|_{C^{2+\alpha;\frac{2+\alpha}{2}}(M_T)} < C$ and so our flow is uniformly smooth. This in turn means that we may continue the flow and leads to long time existence.

These proofs are simple modifications of the elliptic estimates found in Gilbarg and Trudinger [3, Section 17.4]. There is an also a slightly different approach in Lieberman [5, Chapter XIV, section 2] but the fundamental ideas are the same.

As we only need local estimates, throughout this section we will assume that we are working on Balls on the interior a domain $\Omega \times [0, T]$

3.1. **Ingredients.** Throughout we will write $X = (x, t)$ and $Y = (y, s)$. We define the following parabolic rectangles:

$$K(X,R) := \left[\prod_{i=1}^n (x^i - R, x^i + R)\right] \times (t - R^2, t) \subset \Omega \times [0,T]$$

$$\theta(X,R) := K((x, t-4R^2), R) = \left[\prod_{i=1}^n (x^i - R, x^i + R)\right] \times (t - 5R^2, t - 4R^2).$$

We will typically ignore the centerpoint from now on, assuming this to be 0 and writing $K(R), \theta(R)$. The vital piece of machinery needed for our estimates is the following:

Theorem 20 (The Weak Harnack Inequality)**.** *Let $0 \le u \in C^{2,1}(K(5R))$ such that u fulfils the inequality*

$$Lu \equiv -\dot{u} + a^{ij}u_{ij} + b^i u_i \le f$$

on $K(5R)$, then there exist $p(n,\lambda,\Lambda), C(n,\lambda,\Lambda,R\|b\|_{L^\infty(K(5R))}) > 0$ such that

$$\left(R^{-(n+2)}\int_{\theta(R)} u^p\right)^{\frac{1}{p}} \leq C\left(\inf_{K(R)} u + R^{\frac{n}{n+1}}\|f\|_{L^{n+1}(K(5R))}\right)$$

where $0 < \lambda\delta^{ij} \leq a^{ij} \leq \Lambda\delta^{ij}$ as matrices.

The proof (originally by Krylov–Safonov) may be found (in German) in O. Schnürer's lecture notes [9, Theorem 3.21, page 43], see also [5, Theorem 7.37, p187] in Lieberman's book. The notes by S. Picard [7] also look like a good reference. Since we will have to use it repeatedly, we will use the notation

$$\Phi_p[u] = \left(R^{-(n+2)}\int_{\theta(R)} u^p\right)^{\frac{1}{p}}$$

It is important to note that p is not guaranteed to be greater than 1, so the above is not an estimate on a norm. However, in most applications we can get around this using the following replacement for the triangle inequality.

Lemma 21. *If we have measurable functions $f_i \geq 0$, $p > 0$, then*

$$\left(\int\left[\sum_{i=1}^N f_i\right]^p\right)^{\frac{1}{p}} \leq C(N,p)\sum_{i=1}^N\left(\int f_i^p\right)^{\frac{1}{p}}$$

Proof. If $p \geq 1$ we are done. Otherwise:

$$\begin{aligned}
&\left[\int (f_1 + \dots f_N)^p\right]^{\frac{1}{p}} \\
&\leq \Bigg[\int_{\{f_1\geq f_i \forall 1\leq i\leq N\}} f_1^p\left(1+\frac{\sum_{i\neq 1} f_i}{f_1}\right)^p + \dots \\
&\qquad\qquad \dots + \int_{\{f_N\geq f_i \forall 1\leq i\leq N\}} f_N^p\left(1+\frac{\sum_{i\neq N} f_i}{f_1}\right)^p\Bigg]^{\frac{1}{p}} \\
&\leq N\left[\int f_1^p + \dots + \int f_N^p\right]^{\frac{1}{p}} \\
&\leq N^{1+\frac{1}{p}}\max\left\{\left(\int f_1^p\right)^{\frac{1}{p}}, \dots, \left(\int f_N^p\right)^{\frac{1}{p}}\right\} \\
&\leq N^{1+\frac{1}{p}}\sum_{i=1}^N\left(\int f_i^p\right)^{\frac{1}{p}}
\end{aligned}$$

□

Corollary 22. *For all $p > 0$, $f_i \geq 0$*

$$\Phi_p\left[\sum_{i=1}^N f_i\right] \leq C(p,N)\sum_{i=1}^N \Phi_p[f_i]$$

The final requirement will be the following standard lemma:

Lemma 23 (Iteration Lemma). *Suppose σ is a non-decreasing function on $(0, R_0]$, and $0 < \gamma, \tau < 1$, $C > 0$ satisfying*

$$\sigma(\tau R) \leq \tau^\alpha \sigma(R) + R^\alpha C_1$$

for $R \leq R_0$, then

$$\sigma(R) \leq C(\alpha, \tau) \left(\frac{R}{R_0}\right)^\alpha [\sigma(R_0) + C_1] \quad .$$

Proof. Exercise! Alternatively see Gilbarg and Trudinger [3, Lemma 8.23, p201] or Lieberman [5, Lemma 4.6 on p53]. □

We now see how the Weak Harnack inequality may be used to prove Hölder estimates when we have only low regularity on the coefficients of the parabolic equation.

Theorem 24 (Hölder Estimate for Linear Parabolic PDEs). *Let u be a solution of*

$$Lu \equiv -\dot{u} + a^{ij}u_{ij} + b^i u_i = f - du$$

on $K(5R)$. Then for $0 < r \leq R$,

$$\operatorname*{osc}_{K(r)} u \leq c(n, \lambda, \Lambda, \|b\|_{L^\infty}) \left(\frac{r}{R}\right)^\alpha \left(\operatorname*{osc}_{K(R)} u + \|f - du\|_{L^{n+1}(K(5R))} R^{\frac{n}{n+1}}\right)$$

where $\alpha = \alpha(n, \lambda, \Lambda, \|b\|_{L^\infty(K(5R))})$, $0 < \alpha < 1$.

Proof. Take $r \leq R$. We define

$$M_r := \sup_{K(r)} u \ , \qquad m_r := \inf_{K(r)} u \ .$$

Then we have that

$$L(M_{5r-u} - u) = -f + du \ , \qquad L(u - m_r) = f - du$$

where $M_{5r-u} - u > 0$ and $u - m_{5r} > 0$ and so we may apply the Weak Harnack inequality, Theorem 20 to get that there exists a $p > 0$ so that

$$\begin{aligned}
\Phi_p(M_{5r} - u) &\leq C(\inf_{K(r)}(M_{5r} - u) + r^{\frac{n}{n+1}} \|f - du\|_{L^{n+1}(K(5r))}) \\
&\leq C(M_{5r} - M_r + r^{\frac{n}{n+1}} \|f - du\|_{L^{n+1}(K(5r))}) \\
\Phi_p(u - m_{5r}) &\leq C(\inf_{K(r)}(u - m_{5r}) + r^{\frac{n}{n+1}} \|f - du\|_{L^{n+1}(K(5r))}) \\
&\leq C(m_r - m_{5r} + r^{\frac{n}{n+1}} \|f - du\|_{L^{n+1}(K(5r))})
\end{aligned}$$

so putting these together we have

$$\begin{aligned}
M_{5r} - m_{5r} &\leq \Phi_p(M_{5r} - u + u - m_{5r}) \\
&\leq C(p)(\Phi_p(M_{5r} - u) + \Phi_p(u - m_{5r})) \\
&\leq C(M_{5r} - m_{5r} - (M_r - m_r) + 2r^{\frac{n}{n+1}} \|f - du\|_{L^{n+1}(K(5r))})
\end{aligned}$$

So writing $\sigma(r) = \operatorname*{osc}_{K(r)} u = M_r - m_r$ (which is a nonincreasing function) then

$$\sigma(5r) \leq C(\sigma(5r) - \sigma(r) + 2r^{\frac{n}{n+1}} \|f - du\|_{L^{n+1}(K(5r))})$$

or, rearranging,

$$\sigma(r) \leq \frac{C-1}{C}\sigma(5r) + 2r^{\frac{n}{n+1}} \|f - du\|_{L^{n+1}(K(5r)} \ .$$

We may now apply the iteration lemma (Lemma 23) to get the theorem with α the solution to $\left(\frac{1}{5}\right)^{\alpha} = \frac{C-1}{C}$. ☐

Corollary 25 (Hölder estimates for u_t)**.** *Suppose that for some $R_0 < 1$, $u \in C^{4,2}(K(5R_0))$ is a solution of the nonlinear parabolic PDE (23) where F is C^1 and uniformly parabolic at u. Then there exists $0 < \alpha < 1$ and $C_1 > 0$ depending only on n, λ, Λ, $|F_z|_{C^0(\Gamma)}$, $|F_p|_{C^0(\Gamma)}$, $|F_t|_{C^0(\Gamma)}$ and $|u|_{C^{2,1}(K(5R_0))}$ such that for $R < R_0$,*

$$\underset{K(R)}{\operatorname{osc}} u_t \leq C_1 \left(\frac{R}{R_0}\right)^{\alpha} \left(\underset{K(R_0)}{\operatorname{osc}} u_t + R_0^{\frac{n}{n+1}}\right) \quad .$$

Proof. We differentiate the above equation in time, and see that $v = u_t$ satisfies

$$v_t = F^{ij}v_{ij} + F_{p^j}v_j + F_z v + F_t \quad .$$

Apply the previous theorem, Theorem 24, we see that there exists an $\alpha(n, \lambda, \Lambda, |F_p|_{C^0(\Gamma)}) \in (0, 1)$ such that

$$\begin{aligned}\underset{K(R)}{\operatorname{osc}} v &\leq c(n, \lambda, \Lambda, |F_p|_{C^0(\Gamma)}) \left(\frac{R}{R_0}\right)^{\alpha} \left(\underset{K(R_0)}{\operatorname{osc}} v + \|F_t - F_z v\|_{L^{n+1}(K(5R))} R_0^{\frac{n}{n+1}}\right)\\ &\leq c \left(\frac{R}{R_0}\right)^{\alpha} \left(\underset{K(R_0)}{\operatorname{osc}} v + C(n, |F_z|_{C^0(\Gamma)}, |F_t|_{C^0(\Gamma)}, |u|_{C^{2,1}(K(5R_0))}) R_0^{\frac{n}{n+1}}\right)\end{aligned}$$

which gives the Lemma. ☐

We end this section, with the following useful Lemma which will be used to deal with Hölder derivatives of D^2u.

Lemma 26 (Matrix Lemma)**.** *Let $S[\lambda, \Lambda]$ be the set of positive symmetric matrices in $\mathbb{R}^{n\times n}$ with eigenvalues lying in $[\lambda, \Lambda]$. Then there exists a finite set of vectors $\gamma_1, \ldots, \gamma_N \in \mathbb{R}^n$ and $0 < \lambda^* \leq \Lambda^*$ all depending only on n, λ and Λ such that any matrix $A = [a^{ij}] \in S[\lambda, \Lambda]$ may be written in the form*

$$A = \sum_{k=1}^{N} \beta^k \gamma_k \otimes \gamma_k, \ \textit{which implies } a^{ij} = \sum_{k=1}^{N} \beta^k \gamma_k^i \gamma_k^j$$

where $\lambda^* \leq \beta_k \leq \Lambda^*$ for $k = 1, \ldots, N$. Furthermore we may choose $\gamma_1, \ldots \gamma_N$ to inculde the coordinate direction $e_1, \ldots, e_n$ and vectors of the form

$$\frac{1}{\sqrt{2}}(e_i \pm e_j), \ \textit{where } i < j, 1 \leq i, j \leq n \quad .$$

Proof. Writing $\bar{n} = \frac{1}{2}n(n+1)$, we first observe that $S[\lambda, \Lambda]$ is compact in $\mathbb{R}^{\bar{n}}$: We may immediately see that the matrix norm $S^{ij}S_{ij} \leq n\Lambda^2$, so this is a bounded subset of $\mathbb{R}^{\bar{n}}$. The function $\lambda : \mathbb{R}^{\bar{n}} \to \mathbb{R}^n$ which gives the (ordered) eigenvalues is well-known to be continuous. Therefore $S[\lambda, \Lambda) = \lambda^{-1}([\lambda, \Lambda]^n)$ is closed, and so is compact.

For unit vectors $\gamma_i \in \mathbb{R}^n$, $1 \le i \le \bar{n}$, we now write the open set

$$U(\gamma_1, \dots, \gamma_{\bar{n}}) = \left\{ \sum_{i=1}^{\bar{n}} \beta_k \gamma_k \otimes \gamma_k | \beta_k > 0 \text{ for } 1 \le k \le \bar{n} \right\}$$

where we assume $\gamma_1 \otimes \gamma_1, \dots, \gamma_{\bar{n}} \otimes \gamma_{\bar{n}}$ are linearly independant. By diagonalising any $A \in S[\lambda, \Lambda]$, we see that there are unit vectors γ_i such that $A \in U(\gamma_1, \dots, \gamma_{\bar{n}})$. Therefore all such open sets form an open cover of $S[\lambda, \Lambda]$, which has a finite subcover due to compactness. Therefore there exists a finite number of unit vectors $\gamma_1, \dots \gamma_N$ depending only on n, λ, Λ such that

$$A = \sum_{i=1}^{N} \beta_k \gamma_k \otimes \gamma_k \quad , \tag{22}$$

where $\beta_k \ge 0$. Clearly we may add any particular set of unit vectors to the γ_i and the above will still hold.

We now demonstrate that $\beta_k \in [\lambda^*, \Lambda^*]$: Suppose we now take the finite set of vectors such that matrices in $S[\frac{\lambda}{2}, \Lambda]$ may be written as above. Then for any $A \in S[\lambda, \Lambda]$,

$$A' := A - \lambda^* \sum_{i=1}^{N} \gamma_i \otimes \gamma_i \in S[\frac{\lambda}{2}, \Lambda]$$

where we have chosen $\lambda^* = \frac{\lambda}{2N}$. Writing A' as a sum with coefficients $\beta_k' \ge 0$, we see the coefficients of A, $\beta_k \ge \lambda^*$. By considering eigenvalues we see $\beta_k \le \Lambda = \Lambda^*$. □

3.2. **Assumptions on F.** We will require that F is convex, twice differentiable and uniformly parabolic. More explicitly:

Assumption 2. From now on we will assume that:

1. F **is twice differentiable:** $F \in C^2(\Gamma)$, where Γ a convex open subset of $\Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$. We will assume that the solution stays in the subset Γ.
2. F **is uniformly parabolic at u:** We require that F is uniformly parabolic at u, in the sense that $\xi \in \mathbb{R}^n$ there are $0 < \lambda \le \Lambda$ such that
$$0 < \lambda |\xi|^2 \le F^{ij} \xi_i \xi_j \le \Lambda |\xi|^2 \quad .$$
3. F **is concave in D^2u:** We assume that F is a concave function of r_{ij} on the (convex hull of the) range of D^2u. Writing $F^{ij} = \frac{\partial F}{\partial r_{ij}}$, $F^{ij,kl} = \frac{\partial^2 F}{\partial r_{ij} \partial r_{kl}}$ then this in particular implies
   (a) $F^{ij.kl} X_{ij} X_{kl} \le 0$ for all $X_{ij} \in \mathbb{R}^{n\times n}$ and
   (b) $F^{ij}|_{D^2u(y)}(u_{ij}(x) - u_{ij}(y)) \ge F(D^2u(x)) - F(D^2u(y))$.

For $u \in \Gamma_{\lambda,\Lambda}$, we will prove that from uniform $C^{2;1}$ estimates we may obtain uniform $C^{2+\alpha;\frac{2+\alpha}{2}}$ estimates for the equation

$$\begin{cases} Pu = 0 \\ u(\cdot, t) = u_0(\cdot) \end{cases} \quad . \tag{23}$$

3.3. **Interior second derivative Hölder estimates for simple** F**.** In the interest of clarity, we begin by showing our Hölder estimates for the following simpler version of equation (23). We begin by assuming we have a solution $u \in C^{4,2}(K(25R_0))$ (for some $R_0 \leq 1$) to

$$F(D^2u) - u_t = g(X) \tag{24}$$

where we assume that F is twice differentiable, uniformly parabolic at u and convex as in Assumption 2.

From now on, unless otherwise specified, we will take C to be any finite constant depending only on n, λ, Λ, F, DF, R_0, $|g|_{C^{2,1}(K(25R_0))}$ and $|u|_{C^{2,1}(K(25R_0))}$, which may change between lines of the same equation.

As with the previous section we begin by differentiating (24) in (fixed) direction γ. Differentiating once,

$$F^{ij}D_{\gamma ij}u - u_{\gamma t} = g_\gamma$$

and then again

$$F^{ij,kl}D_{\gamma ij}uD_{\gamma kl}u + F^{ij}D_{ij\gamma\gamma}u - u_{t\gamma\gamma} = g_{\gamma\gamma} \quad .$$

We now choose $\gamma_k \in \mathbb{R}^n$, $1 \leq k \leq N$, as in the Matrix Lemma (Lemma 26) where λ and Λ are the parabolicity constants in Assumption 2(2). Writing

$$w^k := u_{\gamma^k\gamma^k}$$

then from the above equation and convexity (condition 3a above) we estimate

$$F^{ij}w^k_{ij} - w^k_t \geq g_{\gamma^k\gamma^k} \quad . \tag{25}$$

We may now apply the Weak Harnack Inequality (Theorem 20). Similarly to in Corollary 25 we define

$$M^k_s = \sup_{K(sR)} w^k, \qquad m^k_s = \inf_{K(sR)} w^k,$$

$$\sigma(sR) = \sum_{k=1}^N \operatorname*{osc}_{K(sR)} w^k = \sum_{k=1}^N M^k_s - m^k_s \qquad \sigma_t(sR) = \operatorname*{osc}_{K(sR)} u_t \quad .$$

Assuming u is defined on $K(5R)$ then we define $v = M_5 - w \geq 0$ and apply the weak Hölder inequality to v which implies there exists $p(n, \lambda, \Lambda) > 0$ and $C_1(n, \lambda, \Lambda, |u|_{C^{2,1}(K(5R))}) > 0$ such that

$$\Phi_p(M^k_5 - w^k) \leq C_1\left(M^k_5 - M^k_1 + R^{\frac{n}{n+1}}\|D^2g\|_{L^\infty(K(5R))}\right) \tag{26}$$

We now aim to get a similar estimate on $\Phi_p(w^k - m^k_5)$ so that we may make the left hand side into an $\operatorname*{osc}_{K(5R)} w$ and then apply the standard iteration lemma (Lemma 23). However, as our equation for w^k, (25) is only an inequality, we will need to use the matrix lemma and concavity to get this.

By the concavity assumption, specifically 3b above,

$$\begin{aligned} u_t(X) - u_t(Y) - F^{ij}|_{D^2u(Y)}(u_{ij}(X) - u_{ij}(Y)) \\ &\leq u_t(X) - u_t(Y) + F(D^2u(Y)) - F(D^2u(X)) \\ &= g(Y) - g(X) \quad . \end{aligned}$$

So by the matrix lemma,

$$u_t(X) - u_t(Y) + \sum_{k=1}^{N} \beta_k [w^k(Y) - w^k(X)] \leq g(Y) - g(X) \tag{27}$$

where $0 < \lambda^* \leq \beta_k \leq \Lambda^*$.

From (27) we have

$$\beta_l(w^l(Y) - w^l(X)) \leq g(Y) - g(X) - u_t(X) + u_t(Y) - \sum_{k \neq l}^{N} \beta_k [w^k(Y) - w^k(X)]$$

and so, maximising the lefthand side over $X \in K(5R)$,

$$w^l(Y) - m_5^l \leq \frac{1}{\lambda^*}\left[5R(|Dg|_{C^0(K(5R))} + 5R|g_t|_{C^0(K(5R))}) + \sigma_t(5R) + \Lambda^* \sum_{k \neq l}^{N} [M_5^k - w^k(Y)]\right]$$

We calculate from (26)

$$\begin{aligned}
\Phi_p(M_5^k - w^k) &\leq C(N,p) \sum_{k \neq l} \Phi_p(M_5^k - w^k) \\
&\leq C\left(\sum_{k \neq l} (M_5^k - M_1^k) + R^{\frac{n}{n+1}} |D^2 g|_{L^\infty(K(5R)}\right) \\
&\leq C\left(\sigma(5R) - \sigma(R) + R^{\frac{n}{n+1}} |D^2 g|_{C^0(K(5R)}\right)
\end{aligned}$$

and therefore (assuming $R < 1$)

$$\Phi_p(w^l - m_5^l) \leq C\left(\sigma(5R) - \sigma(R) + R^{\frac{n}{n+1}} |g|_{C^{2,1}(K(5R))} + \sigma_t(5R)\right) \quad .$$

We now obtain the estimate,

$$\begin{aligned}
\operatorname*{osc}_{K(5R)} w^l &\leq C(p)(\Phi_p(w^l - m_5^l) + \Phi_p(M_5^l - w^l)) \\
&\leq C\left(\sigma(5R) - \sigma(R) + R^{\frac{n}{n+1}} |g|_{C^{2,1}(K(5R))} + \sigma_t(5R)\right)
\end{aligned}$$

Summing over l, we obtain

$$\sigma(5R) \leq C\left(\sigma(5R) - \sigma(R) + R^{\frac{n}{n+1}} |g|_{C^{2,1}(K(5R))} + \sigma_t(5R)\right)$$

or

$$\sigma(R) \leq \delta\sigma(5R) + C\left(R^{\frac{n}{n+1}} |g|_{C^{2,1}(K(5R))} + \sigma_t(5R)\right)$$

where $\delta = \frac{C-1}{C} = \left(\frac{1}{5}\right)^{\hat{\alpha}}$. To deal with the σ_t term, from Corollary 25, for $R_0 < 1$ and $u \in C^{4,2}(K(25R_0))$, then applying Lemma **??**, we see that for $R < 5R_0$, there is an $\alpha > 0$ such that

$$\sigma_t(5R) \leq C\left(\frac{R}{R_0}\right)^{\alpha} \left(\operatorname*{osc}_{K(5R_0)} u_t + R_0^{\frac{n}{n+1}}\right) \quad . \quad .$$

Applying the Iteration Lemma (Lemma 23) we have that for all $R < R_0$,

$$\operatorname*{osc}_{K(R)} u_{\gamma_k\gamma_k} \le C \left(\frac{R}{R_0}\right)^{\tilde{\alpha}} [\operatorname*{osc}_{K(R_0)} u_{\gamma_k\gamma_k} + \operatorname*{osc}_{K(5R_0)} u_t + R_0^{\frac{n}{n+1}}]$$

where $1 \le k \le N$ and $\tilde{\alpha} = \min\{\alpha, \hat{\alpha}\}$. Since in the Matrix lemma, the γ_k contain e_i and $\frac{1}{\sqrt{2}}(e_i \pm e_j)$, we obtain estimates for all elements of D^2u. Summarising the above:

Theorem 27. *Suppose that $u \in C^{4,2}(K(25R_0))$ for some $R_0 < 1$ is a solution of (**??**), then there exists constants $\alpha, C > 0$ depending on n, λ, Λ, R_0, $|g|_{C^{2,1}(K(25R_0))}$ and $|u|_{C^{2,1}(K(25R_0))}$ such that for all $R < R_0$,*

$$\operatorname*{osc}_{K(R)} D^2_{ij}u \le C \left(\frac{R}{R_0}\right)^{\alpha} [\operatorname*{osc}_{K(R_0)} D^2_{ij}u + \operatorname*{osc}_{K(5R_0)} u_t + R_0^{\frac{n}{n+1}}]$$

Remark 3. Versions of the above may be shown under much weaker differentiability assumptions on F, see for example Liebermann, Lemma 14.6 on p366.

Remark 4. Alternatively see the smoothing argument used in Gilbarg and Trudinger in [3, Theorem 17.18].

3.4. **Interior second derivative Hölder estimates for general F.** We prove the following.

Theorem 28. *Suppose that $u \in C^{4,2}(K(25R_0))$ for some $R_0 < 1$ is a solution of (**??**) satisfying conditions 1, 2, 3 above then there exists constants $\alpha, C > 0$ depending only on n, λ, Λ, the $C^0(\Gamma)$ norms of first and second derivatives of F (except F_{rr}), R_0, and $|u|_{C^{2,1}(K(25R_0)}$ such that for all $R < R_0$,*

$$\operatorname*{osc}_{K(R)} D^2u \le C \left(\frac{R}{R_0}\right)^{\alpha} [\operatorname*{osc}_{K(R_0)} D^2u + \operatorname*{osc}_{K(5R_0)} u_t + R_0^{\frac{n}{n+1}}] \quad .$$

In particular this implies

$$\|u\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(K(R))} \le \hat{C}(n, \lambda, \Lambda, F, DF, R_0, |u|_{C^{2,1}(K(25R_0)}) \quad .$$

Proof. As above, we agree write C for any finite constant depending on n, λ, Λ, the $C^0(\Gamma)$ norms of first and second derivatives of F except F_{rr} and $|u|_{C^{2,1}(K(25R_0)}$. We begin by calculating the first and second derivatives of

$$F(X, u, Du, D^2u) - u_t = 0 \quad .$$

First derivatives give

$$F^{ij}u_{ij\gamma} + F_{p^k}u_{k\gamma} + F_z u_\gamma + \gamma^i F_{x^i} - u_{t\gamma} = 0 \tag{28}$$

and the second derivatives

$$
\begin{aligned}
0 &= F^{ij}u_{ij\gamma\gamma} + F^{ij,kl}u_{ij\gamma}u_{kl\gamma} + 2F^{ij}_{p^k}u_{k\gamma}u_{ij\gamma} + 2F^{ij}_{z}u_{\gamma}u_{ij\gamma} + 2\gamma^k F^{ij}_{x_k}u_{ij\gamma} \\
&\quad + F_{p^k}u_{k\gamma\gamma} + F_{p^kp^l}u_{k\gamma}u_{l\gamma} + 2F_{p^kz}u_{k\gamma}u_{\gamma} + 2\gamma^l F_{p^kx^l}u_{k\gamma} \\
&\quad + F_z u_{\gamma\gamma} + F_{zz}u_\gamma u_\gamma + 2\gamma^k F_{zx^k}u_\gamma + \gamma^k\gamma^l F_{x^kx^l} - u_{t\gamma\gamma} \\
&= F^{ij}u_{ij\gamma\gamma} + F^{ij,kl}u_{ij\gamma}u_{kl\gamma} + \left[2F^{ij}_{p^k}u_{k\gamma} + 2F^{ij}_{z}u_\gamma + 2\gamma^k F^{ij}_{x_k} + F_{p^i}\gamma^j\right]u_{ij\gamma} \\
&\quad + F_{p^kp^l}u_{k\gamma}u_{l\gamma} + 2F_{p^kz}u_{k\gamma}u_{\gamma} + 2\gamma^l F_{p^kx^l}u_{k\gamma} \\
&\quad + F_z u_{\gamma\gamma} + F_{zz}u_\gamma u_\gamma + 2\gamma^k F_{zx^k}u_\gamma + \gamma^k\gamma^l F_{x^kx^l} - u_{t\gamma\gamma} \quad .
\end{aligned}
$$

Using concavity, we obtain

$$
F^{ij}u_{ij\gamma\gamma} - u_{t\gamma\gamma} \geq -A^{ij\gamma}u_{ij\gamma} - B_\gamma
$$

where

$$
\begin{aligned}
A^{ij\gamma} =& 2F^{ij}_{p^k}u_{k\gamma} + 2F^{ij}_{z}u_\gamma + 2\gamma^k F^{ij}_{x_k} + F_{p^i}\gamma^j \quad , \\
B_\gamma =& F_{p^kp^l}u_{k\gamma}u_{l\gamma} + 2F_{p^kz}u_{k\gamma}u_\gamma + 2\gamma^l F_{p^kx^l}u_{k\gamma} + F_z u_{\gamma\gamma} + F_{zz}u_\gamma u_\gamma \\
&+ 2\gamma^k F_{zx^k}u_\gamma + \gamma^k\gamma^l F_{x^kx^l} \quad .
\end{aligned}
$$

The only important difference from the special case above is the addition of the $u_{ij\gamma}$ term, and we show that this may be estimated using a small quadratic term in our choice of w^k.

We choose γ_k as in the Matrix Lemma and set $M = \sup_\Omega |D^2u|$ and $h^k = \frac{1}{2}\left(1 + \frac{D_{\gamma_k\gamma_k}u}{1+M}\right)$, so that $0 < h^k < 1$. From the above we see that there exist $A_0, B_0 > 0$ depending on $|u|_{C^{2,1}(\Omega\times[0,T])}$ and (the allowed) first and second derivatives of F such that

$$
F^{ij}h^k_{ij} - h^k_t \geq -\frac{C(n)}{1+M}\left(A_0|D^3u| + B_0\right)
$$

We now set $v = \sum_{k=1}^N (h^k)^2$ and calculate

$$
\begin{aligned}
F^{ij}v_{ij} - v_t &= 2\sum_{k=1}^N F^{ij}h^k_i h^k_j + 2\sum_{k=1}^N (h^kF^{ij}h^k_{ij} - h^k h^k_t) \\
&\geq 2\sum_{k=1}^N F^{ij}h^k_i h^k_j - \frac{C(n,N)}{1+M}\left(A_0|D^3u| + B_0\right) \quad .
\end{aligned}
$$

Due to uniform ellipticity and the choice of the γ_k,

$$
\sum_{k=1}^N F^{ij}h^k_i h^k_j \geq \lambda \sum_{k=1}^N |Dh_k|^2 \geq \frac{\lambda}{4n^3(1+M)^2}|D^3u|^2 \quad .
$$

We now use the v term by setting $w^k = h^k + \epsilon v$ and observe we obtain the differential inequality

$$
\begin{aligned}
F^{ij}w^k_{ij} - w^k_t &\geq \frac{\lambda\epsilon}{4n^3(1+M)^2}|D^3u|^2 - \frac{C(n,N)}{1+M}\left(A_0|D^3u| + B_0\right) \\
&\geq -C(n,N)\left[\frac{B_0}{1+M} + \frac{A_0^2}{\lambda\epsilon}\right] =: -C_0(\epsilon) \quad . \qquad (29)
\end{aligned}
$$

Similarly to the simpler cases above, we set

$$W_s^k = \sup_{K(sR)} w^k, \quad M_s^k = \sup_{K(sR)} h^k, \quad m_s^k = \inf_{K(sR)} h^k, \quad \sigma_t(sR) = \operatorname*{osc}_{K(sR)} u_t$$

and

$$\sigma(sR) = \sum_{k=1}^{N} \operatorname*{osc}_{K(sR)} h^k = \sum_{k=1}^{N} (M_s^k - m_s^k) \quad .$$

We are now ready to apply the Weak Harnack Inequality to $v = W_5^k - w^k$ to give

$$\Phi_p \left[W_5^k - w^k \right] \le C \left(W_5^k - W_1^k + R^{\frac{n}{n+1}} C_0(\epsilon) \right)$$

We additionally observe that

$$\begin{aligned} W_5^k - w^k &\ge M_5^k - h_k - \epsilon \operatorname*{osc}_{K(5R)} \sum_{k=1}^{N} (h^k)^2 \\ &\ge M_5^k - h_k - 2\epsilon\sigma(5R) \end{aligned}$$

where the last inequality follows by estimating

$$\operatorname*{osc}_{K(5R)} (h^k)^2 = \sup_{x,y \in K(5R)} |h^k(x) + h^k(y)||h^k(x) - h^k(y)| \le 2 \operatorname*{osc}_{K(5R)} h^k \quad .$$

We now see

$$\Phi_p \left[M_5^k - h_k \right] \le C \left(W_5^k - W_1^k + \epsilon\sigma(5R) + R^{\frac{n}{n+1}} C_0(\epsilon) \right) \quad . \tag{30}$$

Later, we will also need the following estimate

$$\begin{aligned} \Phi_p \left[\sum_{k \ne l}^{N} (M_5^k - h_k) \right] &\le C \sum_{k \ne l} \Phi_p \left[M_5^k - h_k \right] \\ &\le C \left(\sum_{k \ne l}^{N} (W_5^k - W_1^k) + \epsilon\sigma(5R) + R^{\frac{n}{n+1}} C_0(\epsilon) \right) \\ &\le C \left((1 + 3\epsilon)\sigma(5R) - \sigma(R) + R^{\frac{n}{n+1}} C_0(\epsilon) \right) \end{aligned} \tag{31}$$

where the last line follows by using that for some $z \in K(R)$ where $h^k(z) = M_1^k$ that

$$W_5^k - W_1^k \le M_5^k + \epsilon \sup_{K(5R)} v - M_1^k - \epsilon v(z) \le M_5^k - M_1^k + \epsilon \operatorname*{osc}_{K(5R)} v \quad .$$

As previously we now use concavity to obtain more information. We have that

$$\begin{aligned} F^{ij}(Y,&u(Y), Du(Y), D^2u(Y))(u_{ij}(Y) - u_{ij}(X)) - u_t(Y) + u_t(X) \\ &\le F(Y, u(Y), Du(Y), D^2u(Y)) - F(Y, u(Y), Du(Y), D^2u(X)) - u_t(Y) + u_t(X) \\ &= F(X, u(X), Du(X), D^2u(X)) - F(Y, u(Y), Du(Y), D^2u(X)) \\ &\le D_0|X - Y| \end{aligned}$$

where

$$D_0 = \sup_{X,Y\in K(5R)} \{|F_{x^i}(X, u(X)Du(X), D^2u(Y))| + |F_t(X, u(X)Du(X), D^2u(Y))| + |F_z(X, u(X)Du(X), D^2u(Y))||Du(X)| + |F_p(X, u(X)Du(X), D^2u(Y))||D^2u(X)|\} \ .$$

We observe that D_0 is bounded by the usual quantities. Using the Matrix Lemma,

$$\sum_{k=1}^{N} \beta_k(Y)(h_k(Y) - h_k(X)) = \sum_{k=1}^{N} \beta_k(Y)(D_{\gamma_k\gamma_k}u(Y) - D_{\gamma_k\gamma_k}u(X)) \le CD_0|X-Y| + u_t(Y) - u_t(X) \ , \tag{32}$$

and so minimising the second term on the left hand side we see that for any chosen l,

$$h^l - m_5^l \le \frac{1}{\lambda^*}\left[5D_0R + \Lambda^*\sum_{k\ne k}^{N}(M_5^k - h^k) + \sigma_t(5R)\right] \ .$$

Estimating the summation as in equation (31), we obtain that

$$\Phi_p\left[h^l - m_5^l\right] \le C\left((1+3\epsilon)\sigma(5R) - \sigma(R) + \sigma_t(5R) + 5D_0R + R^{\frac{n}{n+1}}C_0(\epsilon)\right) \ .$$

Now adding this to our earlier estimate (30) we see that

$$M_5^l - m_5^l \le C\left((1+3\epsilon)\sigma(5R) - \sigma(R) + \sigma_t(5R) + 5D_0R + R^{\frac{n}{n+1}}C_0(\epsilon)\right)$$

and summing over l,

$$\sigma(5R) \le C_1\left((1+3\epsilon)\sigma(5R) - \sigma(R) + \sigma_t(5R) + 5D_0R + R^{\frac{n}{n+1}}C_0(\epsilon)\right) \ .$$

Rearranging,

$$\sigma(R) \le \delta(\epsilon)\sigma(5R) + \sigma_t + 5D_0R + R^{\frac{n}{n+1}}C_0(\epsilon)$$

where $\delta = \frac{C_1(1+3\epsilon)-1}{C_1}$. If we now set $\epsilon = \frac{1}{6C_1}$, then $\delta < 1$, and so using the estimates on u_t we have that there exists $C_3, \alpha > 0$ (with the usual dependencies) such that

$$\sigma(R) \le \delta(\epsilon)\sigma(5R) + C_3R^\alpha + 5D_0R + R^{\frac{n}{n+1}}C_0(\epsilon)$$

and we are able to apply the induction lemma to give the Theorem. □

3.5. **Estimates near $t = 0$.** In the previous section we saw that we could get $C^{2+\alpha;\frac{2+\alpha}{2}}$ estimates on the interior of the parabolic domain M_T – therefore by our bootstrapping procedure, we get uniform smooth estimates. However, these decay as $t \to 0$, and so we have no control over the solution in a neighbourhood of $t = 0$. We now fix this issue by assuming higher regularity of the initial data.

A nice reference for the material from this section is [1, Appendix A, page]

Lemma 29. *Suppose that F is twice differentiable, uniformly parabolic and convex, as in Assumption 2. Suppose that $u_0 \in C^4(M)$ and $u \in C^{4,2}(M_T)$ is smooth (but with unknown bounds). There exist constants $C > 0$, $\tau \in (0, \frac{1}{2}]$ depending only on $|u|_{2,M_T}$, $|u_0|_{4,M}$ and F and its derivatives such that for all $x \in M$ and $t \in [0,\tau]$,*

$$|D_{ij}^2u(x,t) - D_{ij}^2u_0(x)| < C\sqrt{t}, \qquad |u_t - F(D^2u_0, Du_0, u_0, x, 0)| < Ct \ .$$

Proof. By replacing $u(x,t)$ with $u(x,t) - u_0(x) - F(D^2u_0, Du_0, u, x, 0)t$ we still obtain an equation of the form $u_t = F(D^2u, Du, u, x, t)$, but our new F is only C^2 (this is where the higher regularity assumption on the initial data comes in). In this proof, I am going to suppose that there is a metric g on M (which doesn't change with time) and write D for the covariant derivatives with respect to g. This makes essentially no difference to the claimed equations. This means that wlog we may assume that $u(\cdot,0) = 0$, $u_t(\cdot,0) = 0$. Differentiating our defining equation twice in space we have that

$$\begin{aligned}
D_k u_t &= F^{ij} D_{ij} D_k u + F_{p^i} D_i D_k u + F_z D_k u + F_{x_k}\\
D_{kl} u_t &= F^{ij} D_{ij} D_{kl} u + F^{ij,ab} D_{ijk} u D_{abl} u + F^{ij}_{p^a} D_{ijk} u D_{al} u + F^{ij}_z D_{ijk} u D_l u + F^{ij}_{x_l} D_{ijk} u\\
&\quad + F_{p^i} D_i D_{kl} u + F^{ab}_{p^i} D_{ik} u D_{abl} u + F_{p^i p^j} D_{ik} u D_{jl} u + F_{p^i z} D_{ik} u D_l u + F_{p^i x^l} D_{ik} u\\
&\quad + F_z D_{kl} u + F^{ij}_z D_k u D_{ijl} u + F_{z p^i} D_k u D_{il} u + F_{zz} D_k u D_l u + F_{z x^l} D_k u\\
&\quad + F_{x_k x_l} + F^{ij}_{x_k} D_{ijl} u + F_{x_k p^i} D_{il} u + F_{x_k z} D_l u\\
&\quad + \text{low order curvature of } g \text{ terms from interchanging covariant derivatives}
\end{aligned}$$

Therefore, estimating all terms that involve only second derivatives, writing $|D^2u|^2 = D^{kl}uD_{kl}u$ where we are raising we have that

$$\begin{aligned}
(|Du|^2)_t &= D^{kl}u(D_{kl}u)_t\\
&\leq D^{kl}uF^{ij}D_{ij}D_{kl}u + D^{kl}uF^{ij,ab}D_{ijk}uD_{abl}u + C_1(|D^3u| + 1)\\
&= F^{ij}D_{ij}|D^2u|^2 - 2F^{ij}D_{ikl}uD^{kl}_j u + C_2(|Du||D^3u|^2 + |D^3u| + 1)\ ,
\end{aligned}$$

However, by uniform parabolicity we have that $F^{ij} > \lambda\delta^{ij}$ and so $2F^{ij}D_{ikl}uD^{kl}_j u > 2\lambda|D^3u|^2$. Therefore $v = |D^2u|^2$ satisfies

$$\begin{aligned}
v_t - F^{ij}D_{ij}v &\leq -(2\lambda - C_2|D^2u|)|D^3u|^2 + C_2(|D^3u| + 1)\\
&\leq -(\lambda - C_2\sqrt{v})|D^3u|^2 + C_3
\end{aligned}$$

where we used Young's inequality of the form $ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$. As v is continuous and $v(\cdot,0) = 0$ there is some $\tau > 0$ (possibly depending on higher derivatives of v) so that on the time interval $[0,\tau]$, $\lambda - C_2\sqrt{v} \leq 0$, and we may take τ to be the maximal such interval. On this time interval we have that $w := v - C_3t \leq 0$ is preserved by the maximum principle. Suppose now that $\tau < \frac{1}{2}$. In this case, we then see that $v(\cdot,\tau) < C_3\tau$ and so $\tau > \frac{\lambda^2}{C_3C_2^2}$ otherwise we contradict the maximality of τ. We now have the first claim.

Exercise: Complete the second claim. Taking a derivative wrt time we have that

$$(u_t)_t = F^{ij}D_{ij}u_t + F_{p^i}D_iu_t + F_zu_t + F_{x_k} \leq F^{ij}D_{ij}u_t + F_{p^i}D_iu_t + C$$

Applying a maximum principle to $u_t - Ct$ (and using that $u_t(\cdot,0) = 0$) we have that $u_t \leq Ct$. A similar estimate with the opposite inequality gives $u_t \geq Ct$. □

Theorem 30. *Suppose that u solves*

$$\begin{cases} u_t = F(D^2u, Du, u, x, t)\\ u(\cdot,0) = u_0(\cdot)\end{cases}$$

on M_T, where we assume that $u \in C^{4,2}(M_T)$ with $u_0 \in C^4(M)$. Suppose additionally that F is twice differentiable, uniformly parabolic and convex, as in Assumption 2. Then there *exists $\tau, C, \alpha > 0$ depending only on $|u_0|^0_{4,M}$ and F (and it's derivatives) such that*

$$|u|_{2+\alpha,M_\tau} < C \ .$$

Proof. Set τ to be as in Lemma 29. Using Lemma 29 and the fact that $|u_0|^0_4 \leq C$, we see that for $r \leq \min\{\sqrt{\tau}, \frac{1}{2}\}$,

$$\sup_{B_r(x)\times[0,r^2]} D^2_{ij}u \leq D^2_{ij}u(x,0) + Cr, \qquad \inf_{B_r(x)\times[0,r^2]} D^2_{ij}u \leq D^2_{ij}u(x,0) - Cr$$

so

$$\operatorname*{osc}_{B_r(x)\times[0,r^2]} D_{ij}u \leq Cr \ . \tag{33}$$

Similarly, we have that

$$\operatorname*{osc}_{B_r(x)\times[0,r^2]} u_t \leq Cr \ .$$

From this we may estimate the R_0 in Theorem 28 any $r \leq \min\{\sqrt{\tau}, \frac{1}{2}\}$ and $x \in M$,

$$[D_{ij}u]_{\alpha, B_r\times[\frac{24}{25}r^2, r^2]} \leq Cr^{1-\alpha} \tag{34}$$

We now consider two points $(x,t), (y,s)$ with $0 \leq s \leq t \leq \tau$ and $x, y \in B_r$. We consider two cases:

Case 1: Suppose that $d((x,t),(y,s)) < \frac{\sqrt{t}}{25}$ **.** Applying Krylov-Safonov in the form of (34) with $r = \sqrt{t}$, we obtain

$$|D_{ij}u(x,t) - D_{ij}u(y,s)| \leq Cd((x,t),(y,s))^\alpha t^{\frac{1-\alpha}{2}} \leq Cd((x,t),(y,s))^\alpha$$

Case 2: Suppose that $d((x,t),(y,s)) \geq \frac{\sqrt{t}}{25}$**.** In this case the oscillation estimate, (33), yields

$$|D_{ij}u(x,t) - D_{ij}u(y,s)| \leq C\sqrt{t} \leq Cd((x,t),(y,s)) \leq Cd((x,t),(y,s))^\alpha$$

Therefore we have that

$$[D^2_{ij}u]_{\alpha,M_\tau} \leq C \ .$$

As $u_t = F(D^2u, Du, u, x, t)$, using properties of compositions of functions as in Lemma 12, we also obtain that $[u_t]_\alpha < C$. As a result of interpolation and the bound on $|u|_0$, the theorem follows. □

Corollary 31. *Suppose that F is twice differentiable, uniformly parabolic and convex, as in Assumption 2. Additionally, suppose that $u_0 \in C^4$ and $|u|_{2,M_T} < C_1$ there exists a constant $C = C(C_1, \lambda, \Lambda)$ such that*

$$|u|_{2+\alpha,M_T} < C \ .$$

Corollary 32. *Suppose that F is l times differentiable, uniformly parabolic and convex, as in Assumption 2. Additionally, suppose that $u_0 \in C^{4+l+\alpha}$ and $|u|_{2,M_T} < C_1$ there exists a constant $C_l = C(C_1, l, \lambda, \Lambda)$ such that*

$$|u|_{4+l+\alpha,M_T} < C_l \ .$$

Essentially, this says that the only thing that leads to the flow not continuing to exist is either the $C^{2,1}$ norm blowing up, or the equation ceasing to be uniformly parabolic, as claimed.

3.6. **Weakening assumptions and alternative proofs.** A natural question is to what extent can we weaken the convexity assumption? The following is known:

- If $n = 2$ then we need no convexity assumption at all. This follows from the (strictly 2 dimensional) theory of quasi-conformal mappings.
- If $n \geq 5$ as least some extra assumption is necessary - there is a counter example.
- If $\hat{F}$ is convex and $F = \hat{F}^p$ for some $p > 0$ and we know that while the flow exists $0 < \delta < F|_u < C$ then the above still holds (e.g. in the case of powers of inverse mean curvature). In the evolution of $u_{\gamma\gamma}$ this adds a difficult looking term of the form

$$p(p-1)\hat{F}^{p-2}\hat{F}^{ij}D_{ij\gamma}u\hat{F}^{kl}D_{kl\gamma}u \leq (1+\epsilon)p(p-1)\hat{F}^{p-2}\hat{F}_\gamma\hat{F}_\gamma + \text{lower order terms} .$$

  However, we may consider the evolution of $u_t = F|_u$. This satisfies a similar evolution (by differentiating the equation in time) but even better, the evolution of F^2 gives a term $-p\hat{F}^{p-1}F^{ij}F_iF_j \leq -p\lambda\hat{F}^{p-1}(\hat{F}_\gamma)^2$. Repeating the arguments in Theorem 28 with w^k replaced with $\hat{w}^k := w^k + \mu F^2$ then give Hölder estimates on $\hat{w}$. However, we already know that F^2 is Hölder continuous and so we get the required result. Writing this out in full is a worthwhile exercise

Another important question: Was the above the best way of proving the Hölder estimates? Possibly not. An alternative route (see e.g. Brendle–Huisken [1, Appendix A]) would be to instead get estimates on u_t as above and then use this to get elliptic estimates on $[D^2u]_\alpha$ from the elliptic equation $F(D^2u, Du, u, x, t) = f := u_t$, for example following the estimates in Gilbarg and Trudinger [3, Chapter 17]. This has the advantage of making the proof of the third bullet point above very quick – you don't need to faff around with extra factors of F.

Appendix A. Miscellaneous referenced theorems

Theorem 33 (Arzelà–Ascoli theorem)**.** *Suppose that (X, d) is a compact metric space and $\{f_i\}_{i\in\mathbb{Z}_{\geq 0}}$ is a sequence of functions, $f_i : X \to \mathbb{R}$, which are equicontinuous on X, that is*

- *there exists a $C > 0$ such that for all $i \in \mathbb{Z}_{\geq 0}$, $|f_i|_0 = \sup_{x\in X} |f_i(x)| < C$, and*
- *for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any $i \in \mathbb{Z}_{\geq 0}$, if $d(x, y) < \delta$ then $|f(x) - f(y)| < \epsilon$.*

Then there exists a subsequence and a continuous function f on X such that $f_{i_j} \to f$ uniformly as $j \to \infty$.

Let V, W be normed linear spaces and suppose that $T : V \to W$. Recall that the operator norm is given by

$$\|T\| := \sup_{\substack{x\in V\\ x\neq 0}} \frac{\|T(x)\|_W}{\|x\|_V} .$$

A mapping is bounded if this is finite.

Theorem 34 (The method of continuity)**.** *Let B be a Banach space and V a normed space, and let L_0, L_1 be bounded maps from B to V. For each $t \in [0, 1]$ set*

$$L_t = (1-t)L_0 + tL_1$$

and suppose that there is a constant C such that

$$\|x\|_B \leq C\|L_tx\|_V$$

for all $t \in [0,1]$. Then L_1 is surjective if and only if L_0 is surjective.

See Gilbarg and Trudinger [3, Theorem 5.2, page 75] for a proof.

Theorem 35 (Contraction mapping theorem). *Suppose that (X, d) is a complete metric space and $\Phi : X \to X$ is a contraction, that is, there exists a $\theta \in [0,1)$ such that for any $x, y \in X$*

$$d(\Phi(x), \Phi(y)) \leq \theta d(x,t) \ .$$

Then Φ has a unique fixed point (i.e. a point $x \in X$ such that $\Phi(x) = x$).

See Gilbarg and Trudinger [3, Theorem 5.1, page 74] for a proof.

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