MAXIMUM PRINCIPLES

1. Exercises

Throughout, we will assume that

- Ω is a domain of compact closure with a smooth boundary
- $u : \overline{\Omega_T} \to \mathbb{R}$ is twice continuously differentiable in space and once continuously differentiable in time with derivatives extending continuously to the boundary (i.e. $u \in C^{2;1}(\overline{\Omega_T})$).
- We define
$$L^0 u = u_t - a^{ij} D_{ij} u - b^i D_i u$$
$$Lu = u_t - a^{ij} D_{ij} u - b^i D_i u - du$$
where we assume both of the above are parabolic linear operators. All coefficients will be assumed to be bounded.

Recall that $\Omega_T := \Omega \times [0, T)$. We define the *parabolic boundary* of Ω_T to be
$$\mathcal{P}\Omega_T = (\partial\Omega \times [0, T)) \cup (\Omega \times \{0\}) \ .$$
The hints provided are one possible way of getting the below results – there are many others.

1.1. The weak maximum principle.

In this subsection, in addition to the assumptions above, suppose that

- L^0, L are weakly parabolic, that is a^{ij} is λ-Λ positive definite with $0 \leq \lambda < \Lambda$.

Exercise 1. Prove the weak maximum principle (version 1): Suppose that $L^0 u \leq 0$ on Ω_T. Then prove that for any $(x, t) \in \Omega_T$,
$$u(x, t) \leq \sup_{\mathcal{P}\Omega_T} u$$
Hints:

1. *Prove the above with $Lu < 0$ (properties of increasing maxima will help you here).*
2. *Extend to $Lu \leq 0$ by modifying u (e.g. consider $u_\epsilon = u + \epsilon e^{-t}$).*

Exercise 2. Prove the weak maximum principle (version 2): Suppose that $Lu \leq 0$ on Ω_T and additionally suppose that $d \leq 0$. Then prove that for any $(x, t) \in \Omega_T$,
$$u(x, t) \leq \max\left\{0, \sup_{\mathcal{P}\Omega_T} u\right\}$$

Exercise 3. Suppose that $f \in C^0(\Omega_T)$. Prove uniqueness of solutions $u \in C^{2;1}(\overline{\Omega_T})$ to $Lu = f$ with Dirichlet boundary conditions for some i.e. Suppose that $Lu = f$ and $Lv = f$ and $u = v$ on $\mathcal{P}\Omega_T$ then $u \equiv v$ on Ω_T.

1.2. **The strong maximum principle.** We now improve on the above - typically, we get these using an extra explicit barrier/comparison function. We will additionally need to assume that:

- L^0, L are parabolic, that is a^{ij} is λ-Λ positive definite with $0 < \lambda < \Lambda$.
- $\partial\Omega$ is smooth (only needed in the Hopf Lemma and Neumann boundary maximum principle).

Exercise 4. Prove the parabolic Hopf Lemma: Suppose that on a cylinder $B_r(0) \times [0, T)$ that there is a point (x_0, t_0) with $t_0 > 0$ and $x_0 \in \partial B_r$ such that

$$u(x_0, t_0) > u(x, t) \text{ for all } (x, t) \in \overline{B_r} \times [0, t_0] \setminus \{(x_0, t_0)\} .$$

Furthermore suppose that $Lu \leq 0$ where either L has $d \leq 0$ or $u(x_0, t_0) = 0$ (and no assumption on d). Then

$$x \cdot Du(x_0, t_0) > 0 .$$

Hints: Start with the case $d \leq 0$.

1. *Consider the function $v = e^{-\alpha|x|^2} - e^{-\alpha r^2}$. Check that $v|_{\partial B_r \times [0,t_0]} = 0$ and, and show that for any $0 < \rho < r$ there exists an $\alpha = \alpha(\rho, L)$ such that $Lv \leq 0$ on $(B_r \setminus B_\rho) \times [0, t_0]$.*
2. *Apply the weak maximum principle to show that there exists an $\epsilon > 0$ such that $u + \epsilon v \leq u(x_0, t_0)$ on $(B_r \setminus B_\rho) \times [0, t_0]$. (Hint: By continuity, we know that there exists a $\delta > 0$ such that on $(B_r \setminus B_\rho) \times \{0\}) \cup (\partial B_\rho \times [0, t_0])$, $u \leq u(x_0, t_0) - \delta$)*
3. *Take derivatives at (x_0, t_0) to get the Lemma.*
4. *For the case with $u(x_0, t_0) = 0$, set $d_+ = \max\{d, 0\}$ and consider $\tilde{L}u := Lu + d_+ u$.*

Exercise 5. Prove the Interior maximum property: Suppose $u \in C^{2;1}(B_r(0) \times [0, T)$ satisfies $Lu \leq 0$. Suppose that a maximum of u occurs at time $t_0 > 0$ at the point $0 \in B_r$ and that either $d \leq 0$ or alternatively $u(0, t_0) = 0$ (and no sign on d). Furthermore, suppose that $u(0, t_0) > u(x, t)$ for all $t < t_0$. Then $u_t(0, t_0) > 0$.
Hints: As previously, start with the case $d \leq 0$.

1. *Consider $v = e^{-(|x|^2 + \alpha(t - t_0))} - 1$, show that for α large enough, $Lv < 0$ on $B_r(0) \times [0, t_0]$.*
2. *Now consider $Q = \{(x, t) \in B_r(0) \times [0, t_0] : |x|^2 + \alpha(t - t_0) \geq 0\}$. Note that on $\mathcal{P}(B_r(0) \times [0, t_0]) \cap Q$, $u < u(0, t_0) - \delta$. Therefore show that (as $v = 0$ on the parabola $\partial Q \setminus (\mathcal{P}(B_r(0) \times [0, t_0]))$) there is an $\epsilon > 0$ so that $u + \epsilon v \leq u(x_0, t_0)$. To do this, you will need a small modification of the above weak maximum principles.*
3. *Therefore deduce the statement by differentiating at $(0, t_0)$.*

Exercise 6. Show that, under the assumptions of the previous step, u cannot be positive. (*Hint, look at the equation...*)

Exercise 7. Prove the Strong Maximum Principle: Suppose that L has $0 < \lambda < \Lambda < \infty$, bounded coefficients and $d \leq 0$. $Lu \leq 0$ for some $u \in C^{2,1}(\Omega_T)$. Suppose that for some $x_0 \in \Omega$ and $t_0 \in (0, T)$

$$\sup_{\Omega_T} u = u(x_0, t_0) .$$

Then $u(x, t) \equiv u(x_0, t_0)$ is a constant function.

Exercise 8. Prove a Neumann maximum principle: For L uniformly parabolic with $d \le 0$ and $u \in C^{2,1}(\Omega_T)$ satisfies

$$\begin{cases} Lu \le 0 & \text{in } \Omega_T \\ Du \cdot \nu \le 0 & \text{on } \partial\Omega \times [0,T) \\ u(\cdot,0) = u_0(\cdot) \end{cases}$$

where ν is the outward pointing unit vector to $\partial\Omega$. Show that $\sup_{\Omega_T} u \le \max\{\sup_\Omega u_0, 0\}$.

1.3. **Nonlinear maximum principle.**

Exercise 9. Prove a nonlinear comparison principle: Suppose that F is C^1 and $P(u) = u_t - F(D^2u, Du, u, x, t)$. Suppose that $u, v \in C^{2,1}(\overline{\Omega_T})$ are admissable and that $\Gamma_{\lambda\Lambda}$ is convex. Suppose that

$$Pu \le Pv \text{ on } \Omega \text{ and } u \le v \text{ on } \mathcal{P}\Omega_T \ .$$

Show that $u \le v$ on Ω_T.
Hint: Use Taylor's theorem/mean value theorem to get a linear parabolic equation for $u - v$.

1.4. **A local maximum principle via the ABP inequality.** For a function u, a point (x,t) is called an upper contact point if there exists a vector ξ such that

$$u(x,t) + \xi \cdot (y - x) \ge u(y,s)$$

for all $(y,s) \in \Omega \times (0,t]$. Define $E(u)$ to be all such upper contact points. Suppose now that $\Omega \subset B_R$. We define

$$E_+(u) = \{(x,t) \in E(u) : R|\xi| < u(x,t) - \xi \cdot x \le \sup u^+\}$$

Exercise 10. Suppose that $u \in C^{2;1}$ and $u \le 0$ on $P(\Omega_T)$. Prove the Alexandrov–Bakelman-Pucci inequality (ABP-inequality):

$$\sup_{\Omega \times (0,T)} u \le c(n) R^{\frac{n}{n+1}} \left(\int_{E_+} |u_t \det D^2 u| \right)^{\frac{1}{n+1}}$$

Hints:

(1) Consider the mapping $\Phi(x,t) = (Du, u - x \cdot Du)$ and compute its Jacobian. Integrate the Jacobian over $E_+(u)$ and estimate this from below by $\int_{\Phi(E_+)} 1 d\mu$.
(2) Next, by considering contact planes show that the sawn-off cone $\Sigma := \{(\xi, h) : R|\xi| < h < \frac{1}{2} \sup u\}$ satisfies $\Sigma \subset \Phi(E_+)$.
(3) The inequality now follows by explicitly computing the volume of Σ in terms of $\sup u$.

Exercise 11. For the simplified parabolic equation $Lu = u_t - a^{ij} D^2_{ij} u$, suppose that $Lu \le f$ on Ω_T. Prove a maximum principle of the form

$$\sup_{\Omega_T} u \le \sup_{\mathcal{P}(\Omega_T)} u + c(n, \lambda, \Lambda) R^{\frac{n}{n+1}} \|f\|_{L^{n+1}} \ .$$

Hint: Use the ABP inequality along with the arithmetic-geometric inequality.