Inverse curvature flows in Riemannian warped products

Habilitationszusammenfassung

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CONTENTS

Preface

This work collects several papers the author Julian Scheuer (J.S.) has written, partially together with other authors, during the last several years. The purpose is the habilitation of J.S. at the university of Freiburg, Germany. All of the attached papers, cf. the appendix, cover various aspects of the theory of inverse curvature flows, which is the main subject J.S. has been working on since he started his Ph.D. thesis. This habilitation thesis is structured as follows.

Chapter 1 provides an overview over the general framework, in which most parts of the attached papers are placed, together with a brief introduction to the broad field of extrinsic curvature flows, as developed during the past 30 to 40 years.

Chapter 2 provides the particular summaries of the presented papers, where we state the main results and highlight the most important techniques which are used. We will also explain how these results fit into the current state of research in their respective subjects. Further information and a broader bibliography on the specific papers can then be found in the appendix itself. According to the habilitation regulations of the university of Freiburg these summaries also contain a detailed description of which author contributed what in case of multiple author papers.

There are two kinds of papers attached to this thesis. Those in Appendices A4 and A5 are newly written works from 2017, which are all submitted for publication in mathematical journals and also appeared on the preprint server arXiv.org. The other appendices consist of three published and two accepted papers. In order to respect copyright regulations, the attached versions are the refereed post-print versions and not the publisher's versions.

Please note that the notation used in the different attached papers may vary, i.e. the same geometric objects may be denoted differently. The reason for this inconvenience is that we eliminated inconsistencies over the years and tried to improve notation. Furthermore, in multiple author papers it is necessary to compromise on a style and hence differences to earlier works occur. Hence the reader is advised to quickly skim the notation and convention sections in each paper.

Acknowledgements

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Chapter 1

INTRODUCTION

This brief introduction is supposed to put the papers presented in this thesis under a general framework, to familiarise the reader with the objects under consideration and to motivate by some examples, why curvature flows are useful and worth to be studied. More detailed introductory comments concerning the specific attached papers will be found in the summaries in Chapter 2 and in the corresponding papers in the appendix.

This thesis deals with the deformation of hypersurfaces within an *ambient manifold* N in direction of a unit normal vector field. The speed of the deformation is determined by several quantities, which are induced by the immersion of the hypersurface. We only consider laws of deformation which can be interpreted as a system of parabolic partial differential equations. For some particular flow speeds, the choices of which are motivated by specific applications to geometric problems, we address the questions of smooth long-time existence, asymptotic behaviour and convergence. We give several applications to geometric questions on hypersurfaces. The precise setting, which subsumes all of the equations dealt with in this thesis, is the following.

Let M be a smooth, connected and orientable manifold of dimension $n \geq 1$, possibly with boundary, and let $(N, \bar{g})$ be a Riemannian manifold of dimension $n+1$. We consider flows of hypersurfaces driven by extrinsic geometric quantities. This means that we are given a time-dependent family of smooth immersions

$$x\colon [0,T) \times M \to N$$

satisfying the equation

$$\begin{aligned} \dot{x} &= -\mathcal{F}(x, \nu, \mathcal{W})\nu \\ x(0, \cdot) &= x_0, \end{aligned} \tag{1.1}$$

where $\dot{x}$ denotes the partial derivative of x with respect to t, $\mathcal{F}$ is strictly monotone with respect to the Weingarten curvature $\mathcal{W}$ and ν is a differentiable normal vector field.[1] In this

[1] To make equation (1.1) parabolic, ν has to be the same normal as given in the Gaussian formula

$$\bar{\nabla}_X Y = x_*(\nabla_X Y) - \sigma h(X, Y)\nu,$$

where h is the second fundamental form of the embedding x.

general setting, the speed $\mathcal{F}$ also depends explicitly on the point $x \in N$ and the variable ν ranges within the unit sphere bundle over the respective immersed hypersurface. The initial embedding x_0 is supposed to be $\mathcal{F}$-admissable, i.e. the domain of $\mathcal{F}$ must contain all the values of $(x, \nu, \mathcal{W})$ along the hypersurface $x_0(M)$. In case that the flow hypersurfaces have a boundary, a further condition shall be placed on it, such as a Dirichlet or Neumann-type condition.

The most famous example of the flow (1.1) is the *mean curvature flow* with

$$\mathcal{F} = H = \operatorname{tr}(\mathcal{W}).$$

The one-dimensional case of the mean curvature flow is also called *curve shortening flow.* The investigation of these flows for $N = \mathbb{R}^{n+1}$ originated in the papers of Brakke [9], Huisken [60], Gage [34], Gage/Hamilton [36] and Grayson [52] and they have received enormous attention from then on. The optimal prototype kind of result is, roughly speaking, that the flow contracts an initially embedded, general hypersurface to a point and after a blow up to unit volume the rescaled hypersurfaces converge to a round sphere. For the mean curvature flow with $n \geq 2$ this result holds true in case that the embedded hypersurface is convex, [60], and in case $n = 1$ the embeddedness is sufficient, [52]. After these seminal works, numerous similar results were achieved for other functions $\mathcal{F} = \mathcal{F}(\mathcal{W})$ of the Weingarten curvature satisfying some (more or less expected) assumptions. Among the major pioneers in this area are, besides the ones already mentioned, Ben Andrews, Bennett Chow, Claus Gerhardt, Richard Hamilton, Kaising Tso and John Urbas, e.g. [3, 4, 18, 19, 40, 57, 58, 102, 103, 104]. It is way beyond the scope of this thesis to make an attempt to give a complete overview over the literature, so let us start to focus on the particular flows which are relevant here.

The papers presented in this thesis all deal with curvature flows of *inverse type,* namely positive curvature will tend to drive the hypersurface outward. The prototype is the so-called *inverse mean curvature flow* (IMCF), with

$$\mathcal{F} = -\frac{1}{H}.$$

Among other negative speeds, this flow was studied in [40, 103, 104] for initial embeddings of starshaped and mean-convex ($H > 0$) hypersurfaces in $\mathbb{R}^{n+1}$. The result in these papers is that the flow exists for all times and after blow-down to unit volume converges to a round sphere. Similarly to the contracting flows, also for expanding flows there appeared many variants and generalisations after these seminal papers, namely generalisations to other ambient spaces and other speed functions $\mathcal{F}$. For a quite extensive (but far from complete) list of references we refer to the introduction and bibliography of Appendix A4. Also inverse curvature flows with boundary conditions have been considered, especially in recent years, cf. [82] and [72, 73][2]. The latter two works are the content of Appendices A6 and A7.

The IMCF has proven to be highly powerful in the deduction of geometric inequalities of hypersurfaces. For instance, the monotonicity of the Hawking mass

$$m_H(M_t) = \frac{|M_t|^{\frac{1}{2}}}{(16\pi)^{\frac{3}{2}}} \left(16\pi - \int_{M_t} H^2\right)$$

for connected flowing hypersurfaces $M_t = x(t, M)$ in an ambient 3-manifold of non-negative scalar curvature under a smooth IMCF was already observed by Geroch [48] and proposed as a tool to prove the positive mass theorem. However, if the initial hypersurface is not

[2] A red bibliography link indicates J.S. to be author or coauthor.

starshaped, the IMCF will in general produce finite time singularities. It was the great achievement of Huisken and Ilmanen [63] to define a new notion of solution to IMCF using a variational principle for the corresponding level-set flow. This so-called *weak inverse mean curvature flow* jumps over the singularities, while the monotonicity of the Hawking mass is preserved. In consequence they could prove the Riemannian Penrose inequality in asymptotically flat 3-manifolds of non-negative scalar curvature, implying the positive mass theorem, which however was first proved by Schoen and Yau [98] by other methods.

Also more general inverse curvature flows of the form

$$\dot{x} = \frac{1}{F}\nu$$

have been useful in the deduction of geometric inequalities. For instance, Guan and Li [53] used them with

$$F = n\frac{H_k}{H_{k-1}},$$

where H_k is the k-th normalised elementary symmetric polynomial, to generalise the Alexandrov-Fenchel quermassintegral inequalities, classically known for convex hypersurfaces, to starshaped and F-admissable (k-convex) hypersurfaces of $\mathbb{R}^{n+1}$. They read

$$\frac{1}{|\mathbb{S}^n|}\int_M H_{k+1} \geq \left(\frac{1}{|\mathbb{S}^n|}\int_M H_k\right)^{\frac{n-k-1}{n-k}} \tag{1.2}$$

with equality precisely at spheres. Similar inequalities for hypersurfaces of the hyperbolic space and the sphere were proven using inverse curvature flows in [27, 37, 107, 110] and [50, 110], [81] respectively. In more general spaces such applications were given in [13, 39, 108].

Until now we have only described flows which depend solely on the Weingarten operator $\mathcal{W}$. However, one can also incorporate dependencies on the normal and on the position vector to solve geometric problems using a flow. For example consider the following *Minkowski problem*. Given a function $f = f(x,\nu)$ defined in the ambient space and the unit sphere bundle of N, and a curvature function $F = F(\mathcal{W})$, one can search for an embedding

$$x\colon M^n \to N^{n+1},$$

such that the F-curvature of that hypersurface $x(M)$ equals f,

$$F_{|x(M)} = f.$$

Under the assumption of barriers and further suitable assumptions on the data of the problem, one can show that the flow (1.1) with the speed

$$\mathcal{F} = F - f$$

converges to a solution. Such problems were for example treated in [41, 42, 44]. Other applications of flows with dependencies on x and ν can be found for example in [14], [65, 66, 112]. There are many other useful flows with such general dependencies. Instead of going into more detail in this introduction, we especially refer to Section 2.3 for more background. The paper in Appendix A5 proves a convergence result for a flow of this kind, together with an application to a geometric inequality.

These examples should provide some motivation to analyse inverse curvature flows and search for applications of them. A detailed description of the theorems proven in this thesis will appear in the following summaries in Chapter 2.

Chapter 2

SUMMARIES

2.1 Appendix A1

The Appendix A1 "Isotropic functions revisited" reviews and extends the most relevant properties of the curvature functions $\mathcal{F}$ used in (1.1), with particular emphasis on the dependence of the curvature variable $\mathcal{W}$. As we have already seen in the introduction, one may think of this function as a function of the principal curvatures, such as for the mean curvature,

$$\mathcal{F} = H = \sum_{i=1}^{n} \kappa_i,$$

or simply as the trace of the Weingarten operator,

$$H = \operatorname{tr}(\mathcal{W}).$$

It is then natural to ask if, given any function $f\colon \mathbb{R}^n \to \mathbb{R}$ of the principal curvatures

$$f = f(\kappa_1, \ldots, \kappa_n)$$

and a real n-dimensional vector space V, do we obtain a function F defined on a suitable subset of the linear transformations $\mathcal{L}(V)$ of V, such that

$$F(A) = f(\mathrm{EV}(A)),$$

where $\mathrm{EV}(A)$ is the n-tuple of eigenvalues of A. Of course, to make this well-defined we have to impose symmetry on f, since there is no preferred ordering of $\mathrm{EV}(A)$. Furthermore, we have to assume that A ranges in the space of self-adjoint endomorphisms with respect to a scalar product g, in order to achieve that A is diagonalisable with $\mathrm{EV}(A) \in \mathbb{R}^n$. Hence for different g we actually get different F, which we temporarily denote by

$$\begin{aligned} F_g\colon \Sigma_g(V) &\to \mathbb{R} \\ A &\mapsto f(\mathrm{EV}(A)), \end{aligned}$$

where $\Sigma_g(V) \subset \mathcal{L}(V)$ denotes the subspace of g-selfadjoint operators. Since the value $f(\mathrm{EV}(\mathrm{A}))$ is invariant under a general linear isometry, a natural associated operator F must satisfy

$$F_g(A) = F_{\tilde{g}}(TAT^{-1}) \tag{2.1}$$

for all orthogonal linear transformations $T\colon (V,g) \to (\tilde{V},\tilde{g})$ between arbitrary inner product spaces. We require such a natural function F, since in the applications to curvature flows the argument $\mathcal{W}$ (the Weingarten operator) formally ranges in varying vector spaces and is self-adjoint with respect to varying inner products (namely the induced metrics). However, using the condition (2.1) we see that F is already determined by its values on the space of symmetric, real $n \times n$ matrices. From this point of view, the story could be considered to be over, since one could always use an orthonormal basis to reduce the action of F on a arbitrary linear transformation to its action on a symmetric matrix. However, in the application to curvature flows, this would mean that we always have to pick a local orthonormal moving frame (e_i), in order to make sense of an expression like

$$\frac{d}{dt}_{|t=0} F(\mathcal{W}(t)) = \lim_{t\to 0} \frac{F(\mathcal{W}(t)) - F(\mathcal{W}(0))}{t} = dF(\mathcal{W})\dot{\mathcal{W}}, \tag{2.2}$$

when dealing with a time dependent argument $\mathcal{W}(t)$. Note that due to the naturality of F the two middle expressions make perfect sense, although $\mathcal{W}(t)$ lies in varying inner product spaces $(V(t), g(t))$. But the expression on the right hand side only makes sense, if the chain rule is applicable, and therefore we need a uniform domain of definition for F. After pulling back to the symmetric matrices, such a uniform domain of definition is available, but this requires the choice of a local moving orthonormal frame. However, it is often most convenient to work in a simple coordinate frame given by the embedding, and then using the expression on the right hand side is troublesome.

The article in Appendix A1 provides a new approach, how to obtain an associated operator F defined, for each vector space V, on the whole space of linear transformations $\mathcal{L}(V)$, which for each inner product g on V coincides with the function F_g on $\Sigma_g(V)$. For a family of immersions

$$x\colon (0,T) \times M \to N$$

this will imply, that we can view F to be defined on the endomorphism bundle $T^{1,1}(M)$ of the manifold M, with the additional property that in any local trivialisation of $T^{1,1}(M)$ with coordinates (x^k, h^i_j) there holds

$$\frac{\partial F}{\partial x^k} = 0.$$

In this sense, F is only defined on the Weingarten operator $F = F(h^i_j)$ and we denote

$$F^i_j = \frac{\partial F}{\partial h^j_i}.$$

In the following theorem the set $\mathcal{D}_\Gamma(V)$ denotes those real diagonalisable endomorphisms of V with eigenvalues in a symmetric set $\Gamma \subset \mathbb{R}^n$. The main results of Appendix A1 can be summarised as follows.

2.1.1 Theorem. *Let V be an n-dimensional real vector space. Let $\Gamma \subset \mathbb{R}^n$ be open, invariant under the permutation group and $f \in C^\infty(\Gamma)$ be symmetric. Then there exists an open set $\Omega \subset \mathcal{L}(V)$ and a smooth function $F \in C^\infty(\Omega)$, such that*

$$F_{|\mathcal{D}_\Gamma(V)} = f \circ \mathrm{EV}_{|\mathcal{D}_\Gamma(V)}. \tag{2.3}$$

Note that there is no claim of uniqueness, but there is a canonical choice for F, which also provides a simple proof of Theorem 2.1.1. Namely, given a symmetric $f \in C^{\infty}(\Gamma)$ a classical result due to Glaeser [51] says that f is a function of the elementary symmetric polynomials s_k,

$$f = \rho(s_1, \dots, s_n), \tag{2.4}$$

where ρ is smooth. Since the s_k satisfy Theorem 2.1.1 with associated operator function

$$S_k(A) = \frac{1}{k!}\frac{d^k}{dt^k}\det(\mathrm{id} + tA)_{|t=0},$$

the natural candidate for F is

$$F = \rho(S_1, \dots, S_n) \tag{2.5}$$

and this F certainly satisfies (2.3). The major contribution of Appendix A1 is to deduce relations of the derivatives of f and F in *any* direction and not only in self-adjoint directions, as it was done in previous works on this topic, e.g. [6, 43]; We obtain:

2.1.2 Theorem. *Let f and F be as in (2.4) and (2.5) and let $A \in \mathcal{D}_{\Gamma}(V)$.*

(i) The derivative of F at A satisfies

$$dF(A)B = \mathrm{tr}(F'(A) \circ B) \quad \forall B \in \mathcal{L}(V),$$

where $F'(A) \in \mathcal{L}(V)$ is a suitable linear map. The maps A and $F'(A)$ are simultaneously diagonalisable and, for a basis (e_i) of eigenvectors for A with eigenvalues (κ_i), the corresponding eigenvalue $F^i(A)$ of $F'(A)$ is given by

$$F^i(A) = \frac{\partial f}{\partial \kappa_i}(\kappa).$$

(ii) Let (η^i_j) be a matrix representation of some $\eta \in \mathcal{L}(V)$ with respect to the basis (e_i). Then the second derivative of F satisfies

$$d^2F(A)(\eta, \eta) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial \kappa_i \kappa_j}\eta^i_i \eta^j_j + \sum_{i \neq j}^{n} \frac{\frac{\partial f}{\partial \kappa_i} - \frac{\partial f}{\partial \kappa_j}}{\kappa_i - \kappa_j}\eta^i_j \eta^j_i, \tag{2.6}$$

where the latter quotient is also well defined if $\kappa_i = \kappa_j$.

In symmetric directions (η^i_j), (2.6) was derived in [6, 43]. Note that Appendix A1 also provides new, quite elementary proofs of the corresponding formulae in this special case. The main reason for us to prove this more general version was a special need for this formula arising in the paper [15], where it remained unclear, whether under a particular curvature flow the time dependent Weingarten operators $\mathcal{W}(t)$ satisfy $\dot{\mathcal{W}}(t) \in \Sigma_{g_t}(T^{1,0}(M))$. Hence an application of the chain rule as in (2.2) was (in our setup using coordinate frames) unjustified. In Appendix A1, some further useful estimates for dF are deduced, when f satisfies some commonly used properties, such as inverse concavity. These are especially relevant to curvature flows. For further details and more introductory comments we refer directly to Appendix A1, especially Section 4.

2.2 Appendices A2 and A3

In this section we summarise the Appendices A2 and A3, "Pinching and asymptotical roundness for inverse curvature flows in Euclidean space" and "Explicit rigidity of almost-umbilical hypersurfaces". They are partially related, as we will see soon. Appendix A2 deals with expanding curvature flows of the form (1.1) in $\mathbb{R}^{n+1}$ with a particular structure:

$$\dot{x} = \frac{1}{F^p}\nu, \quad 0 < p < \infty, \tag{2.7}$$

where F arises from a symmetric function $f \in C^\infty(\Gamma)$, which is positive, strictly monotone, 1-homogeneous and concave and satisfies

$$f_{|\partial\Gamma} = 0.$$

Compare Appendix A1, Section 4, for more details on these properties. $\Gamma \subset \mathbb{R}^n$ is an open, convex and symmetric cone containing the positive cone Γ_+. In case $p > 1$ it is also assumed that $\Gamma = \Gamma_+$. For $p = 1$, long-time existence, convergence to infinity and convergence of suitably rescaled hypersurfaces to a sphere was proven in the seminal papers [40, 103]. For $p \neq 1$ the same result was obtained in [46] by Gerhardt, with earlier results in some particular cases by other authors [20, 21, 68, 76, 97].

The aim of Appendix A2 is to obtain a refined description of the asymptotic behaviour of solutions to (2.7). The motivation is the following. Suppose we have a sphere $\mathcal{S} = \partial B$ around $x \in \mathbb{R}^{n+1}$, pick a point $z \in B$, $z \neq x$, and write

$$u_z(y) = |z - y|.$$

Starting (2.7) from the initial hypersurface $\mathcal{S}$, the flow hypersurfaces M_t are also spheres around x, although the time dependent graph functions $u = u(t, y)$ have a uniformly positive oscillation and hence do not reflect the spherical shape of the flow hypersurfaces. Only after the rescaling with a suitable scaling factor Θ,

$$\tilde{u}_z = \Theta^{-1} u_z,$$

these oscillations are killed and $\tilde{u}_z$ converges to a constant. It is a natural question, if we can optimise the graphical parametrisation, i.e. find the center x and investigate how strongly the new parametrisations $\tilde{u}_x$ converge. The following theorem is the main result of Appendix A2. It shows that even without rescaling, the solutions to the flow (2.7) fit arbitrarily close to a flow of spheres around a uniquely determined optimal center.

2.2.1 Theorem. *Let $n \geq 2$, $0 < p < \infty$, $\Gamma \subset \mathbb{R}^n$ be an open, symmetric and convex cone containing the positive cone and let F be associated to a positive, symmetric, strictly monotone, 1-homogeneous and concave curvature function $f \in C^\infty(\Gamma)$ with $f_{|\partial\Gamma} = 0$. In case $p > 1$ suppose that $\Gamma = \Gamma_+$. Let*

$$x_0 \colon M \hookrightarrow M_0 \subset \mathbb{R}^n$$

be a smooth embedding of a closed and F-admissable hypersurface, which can be written as a graph over a sphere $\mathbb{S}^n$,

$$M_0 = \{(u(y), y) \colon y \in \mathbb{S}^n\}.$$

Then for the unique solution $M_t = x(t, M)$ of (2.7) with initial embedding x_0 there exists a point $Q \in \mathbb{R}^{n+1}$ and a sphere $S^* = S_{R^*}(Q)$ around Q with radius R^*, such that the spherical solutions S_t with radii R_t of (2.7) with initial data S_{R^*} satisfy

$$\operatorname{dist}(M_t, S_t) \leq cR_t^{-\frac{p}{2}} \quad \forall t \in [0, T^*), \tag{2.8}$$

$c = c(p, M_0, F)$. Here dist *denotes the Hausdorff distance of compact sets.*

Note that the long-time existence and rescaled convergence of the flows under consideration was proved in [46]. The point here is to find a flow of spheres which satisfies (2.8). Similar result for the inverse Gauss curvature flow with $n = 2$ were obtained in [97] and for the expansion of curves in $\mathbb{R}^2$ in [77].

Although the technical details are delicate, the idea for the proof of Theorem 2.2.1 can be summarised quickly. The evolution equation of the traceless second fundamental form,

$$\mathring{A} = A - \frac{1}{n}Hg,$$

has a structure that suggests strong stability at totally umbilical hypersurfaces. This means, that once

$$\|\mathring{A}\|^2 = \|A\|^2 - \frac{1}{n}H^2$$

is uniformly small it will decay strongly to 0. Computational details on these claims can be found in Appendix A2, Prop. 3.4. A particular closeness to zero can be deduced from [46]. Strong decay to zero then follows. The second ingredient is a quantitative version of the so-called *umbilicity theorem* (german: *Nabelpunktsatz*), which states that a hypersurface M satisfies

$$\mathring{A} = 0 \quad \Rightarrow \quad M \text{ is a sphere.}$$

We need a quantitative version in the sense of

$$\|\mathring{A}\| < \delta \quad \Rightarrow \quad \operatorname{dist}(M, S_R) < \epsilon,$$

with explicit dependence of δ on ϵ. A paper by Kurt Leichtweiß, [74], provides such an estimate in the class of convex hypersurfaces (which suffices for our purposes) and hence we can conclude, for each time t, the existence of a centre Q_t and a radius R_t, such that

$$\operatorname{dist}(M_t, S_{R_t}(Q_t)) \sim \epsilon_t.$$

The ϵ_t are given explicitly in Appendix A2. It is left to show that the centres Q_t converge in $\mathbb{R}^{n+1}$, which can easily be accomplished. All details are given in Appendix A2.

The power of the quantitative umbilicity theorem in this application was the main reason for J.S. to get deeper into the theory of pinching theorems for hypersurfaces. Generally, such results are lead by the following philosophy: Suppose we have a geometric quantity on a hypersurface, which takes a particular value precisely on a geodesic sphere or another distinguished object. We then ask, whether the hypersurface is close to this distinguished object, provided the geometric quantity is close to the particular value. This game can be played with various quantities, such as the traceless second fundamental form as above, the traceless Einstein tensor, e.g. [26], the first eigenvalue of the Laplace-Beltrami operator on an isometrically embedded Riemannian manifold, e.g. [22, 93] or [94]. Also one can vary the sense of closeness that is asked for (Hausdorff-distance, isometric deficit) under various pinching conditions on the geometric quantities.

Appendix A3, jointly written with Julien Roth, deals with a pinching result of almost-umbilical type. Roughly stated, it says that an immersed hypersurface in $\mathbb{R}^{n+1}$ with a sufficiently small L^p norm of the traceless second fundamental form is Hausdorff close and quasi-isometric to a sphere, provided $p > n$. The precise formulation is as follows.

2.2.2 Theorem. *Let $M \hookrightarrow \mathbb{R}^{n+1}$ be a closed, connected, oriented and immersed C^2-hypersurface with $|M| = 1$. Let $p > n \geq 2$. Then there exist constants $c, \epsilon_0 > 0$ depending on n, p and $\|A\|_p$, as well as a constant $\alpha = \alpha(n,p)$, such that whenever there holds*

$$\|\mathring{A}\|_p < \|H\|_p \epsilon_0,$$

then

$$d_{\mathcal{H}}(M, S_R(x_M)) \leq \frac{c^\alpha R}{\|H\|_p^\alpha} \|\mathring{A}\|_p^\alpha \equiv R\epsilon^\alpha$$

and M is ϵ^α-quasi-isometric to a sphere S_R with a certain radius R.

By ϵ^α-quasi-isometric we mean that a suitable diffeomorphism F from M into S_R satisfies

$$|d(F(x_1), F(x_2)) - d(x_1, x_2)| \leqslant R\epsilon^\alpha$$

for any $x_1, x_2 \in M$.

This theorem is particularly useful, when it is not a priorily clear that M is strictly convex, as it was the case in the application to Appendix A2. For example it was used by Wei in [109] in the proof of a Minkowski type inequality in Schwarzschild space. For surfaces ($n = 2$), similar results were originally proved by De Lellis and Müller even in the critical case $p = 2$, [24, 25]. Also compare the comprehensive description in [89]. The proof of Theorem 2.2.2 is an application of another pinching result J.S. has published together with Julien Roth [94]. Let us give a simplified shortcut of an explanation here.

For an immersed hypersurface $M \subset \mathbb{R}^{n+1}$, Reilly [90] has obtained an upper bound for the first positive eigenvalue of the Laplace-operator on M,

$$\lambda_1(M) \leq \frac{n}{|M|} \int_M H^2 \, d\mu \tag{2.9}$$

with equality precisely if M is a sphere. In [94] we prove that (up to some technical adjustments) if (2.9) almost holds in a sense that will be apparent from the proof of Thm. 1.1 in Appendix A3, then M must be close to a sphere. Hence it would suffice to almost provide (2.9) in order to prove Theorem 2.2.2. A connection between $\lambda_1(M)$ and the umbilicity tensor is provided by the Gauss equation, which links the Ricci tensor of the induced metric on M to the second fundamental form, and a theorem due to Aubry [8], which says that for $p > n/2$, a complete Riemannian manifold (M^n, g) with

$$\frac{1}{|M|} \int_M (\underline{\mathrm{Ric}} - (n-1))_-^p < \frac{1}{C(p,n)}$$

is compact and satisfies

$$\lambda_1 \geqslant n \left(1 - C(n,p) \left(\frac{1}{|M|} \int_M (\underline{\mathrm{Ric}} - (n-1))_-^p \right)^{\frac{1}{p}} \right),$$

where $\underline{\mathrm{Ric}}$ denotes the smallest eigenvalue of the Ricci tensor and

$$x_- = \max(0, -x).$$

A simple calculation gives the desired pinching estimate on $\lambda_1(M)$. Details can be found in Appendix A3, proof of Thm. 1.1.

Section 1 and 2 of Appendix A3, which also include the complete proof of Theorem 2.2.2, were elaborated by both Julien Roth and J.S. to an equal amount.

Section 3 of Appendix A3, **which was elaborated and written by J.S.**, is a straightforward transfer of Theorem 2.2.2 to conformally flat ambient manifolds, based on the observation that the umbilicity tensors of the same hypersurface with respect to two conformally equivalent ambient metrics only differs by the conformal factor.

The final section of Appendix A3, **due to J.S.**, is highly related to the spherical closeness estimates in Appendix A2. It provides an optimality result for the previously mentioned pinching result due to Leichtweiß, [74], namely we prove that an estimate of the form

$$d_{\mathcal{H}}(M, S_R(x_0)) \le c\|\mathring{A}\|_\infty^\alpha, \quad \alpha > 1$$

is not possible in the class of C^∞-bounded hypersurfaces. Precisely:

2.2.3 Theorem. *Let $n \ge 2$ and $C = 2\max(|S_2(0)|, \|\bar{A}_{S_2}\|_\infty)$. For all $\alpha > 1$ and for all $k \in \mathbb{N}$ there exists a uniformly convex smooth hypersurface $M_k \hookrightarrow \mathbb{R}^{n+1}$ with*

$$\max(\|A_k\|_\infty, |M_k|) \le C,$$

such that

$$\|\mathring{A}_k\|_\infty < \frac{1}{k}$$

and for all spheres $S \subset \mathbb{R}^{n+1}$ there holds

$$d_{\mathcal{H}}(M_k, S) > k\|\mathring{A}_k\|_\infty^\alpha.$$

Here $\bar{A}_{S_2}$ denotes the second fundamental form of the sphere with radius 2.

The idea to prove Theorem 2.2.3 uses a counterexample due to Mu-Tao Wang and Pei-Ken Hung [64] concerning the inverse mean curvature flow in the hyperbolic 3-space $\mathbb{H}^3$ (we restrict this description to the case $n = 2$ for simplicity). They prove the existence of an initial starshaped and mean convex hypersurface $M_0 \subset \mathbb{H}^3$ such that the rescaled metrics

$$\tilde{g}_t = \frac{g_t}{|M_t|}$$

do not converge to the round metric. This is tantamount to saying that, after representing M_0 as a graph over a geodesic sphere by a function u, the corresponding graph functions $u(t,\cdot)$ along IMCF do not converge to a constant after rescaling,

$$\hat{u} = u - \frac{t}{2} \to \hat{u}_\infty \ne \text{const}\,.$$

However, in [95] it was deduced that

$$\|\mathring{A}\|_\infty \le ce^{-t}$$

along IMCF in $\mathbb{H}^3$. Switching to the conformally flat parametrisation of $\mathbb{H}^3$ and denoting the Euclidean quantities by a tilde, we obtain

$$\|\mathring{\tilde{A}}\|_\infty \le ce^{-\frac{t}{2}}.$$

Assuming that Theorem 2.2.3 is false, one gets a strong roundness estimate for the $\tilde{M}_t \subset \mathbb{R}^3$. Using arguments similar to those in Appendix A2, Section 4, it is possible to deduce $\hat{u}_\infty = $ const, a contradiction. Complete details are presented in Appendix A3, Section 4.

2.3 Appendices A4 and A5

In these appendices we investigate inverse curvature flows in Riemannian warped products. Appendix A4 "Inverse curvature flows in Riemannian warped products" deals with flows only depending on curvature, where the flow leaves move to the unbounded part of the warped product, whereas Appendix A5 "Locally constrained curvature flows" involves flows with dependencies on the normal and the radial direction, which force the flow to remain in a compact region. Let us first describe Appendix A4. Here we consider the flow

$$\dot{x} = \frac{1}{F^p}, \quad 0 < p \leq 1,$$

where F is the associated operator function to a positive, symmetric, strictly monotone, 1-homogeneous and concave curvature function $f \in C^\infty(\Gamma)$ on a symmetric, open and convex cone $\Gamma \subset \mathbb{R}^{n+1}$ containing the positive cone Γ_+. The ambient manifold is the Riemannian warped product

$$N = (R_0, \infty) \times \mathcal{S}_0, \quad \bar{g} = dr^2 + \vartheta^2(r)\sigma,$$

where $(\mathcal{S}_0, \sigma)$ is a smooth, compact and orientable Riemannian manifold of dimension $n \geq 2$ and $\vartheta \in C^\infty((R_0,\infty))$ satisfies $\vartheta' > 0$ and $\vartheta'' \geq 0$. Further specific assumptions on ϑ and F will appear, depending on how general we assume σ to be. The long-time existence of a solution can be proved in a quite general setting, but in order to get good asymptotic estimates, we will need some control on the derivatives of ϑ, namely we will impose the following assumption.

2.3.1 Assumption. Assume the warping function $\vartheta \in C^\infty((R_0,\infty))$ to satisfy

$$\limsup_{r\to\infty} \frac{\vartheta''\vartheta}{\vartheta'^2} < \infty \quad \text{and} \quad \limsup_{\substack{r\to\infty \\ \vartheta''(r)>0}} \frac{\vartheta'''\vartheta}{\vartheta'\vartheta''} < \infty.$$

In the following theorem $\underline{\widehat{\mathrm{Rc}}}$ denotes the smallest eigenvalue of the Ricci tensor of σ and H_k denotes the curvature function determined by the k-th normalized elementary symmetric polynomial of the principal curvatures. The following theorem is the main result of Appendix A4.

2.3.2 Theorem. *Let $(\mathcal{S}_0, \sigma)$ be a smooth, compact and orientable Riemannian manifold of dimension $n \geq 2$, $R_0 > 0$, $N = (R_0,\infty) \times \mathcal{S}_0$ and define a warped product metric on N,*

$$\bar{g} = dr^2 + \vartheta^2(r)\sigma,$$

with $\vartheta \in C^\infty((R_0,\infty))$, $\vartheta'' \geq 0$ and $\vartheta' > 0$. Let $0 < p \leq 1$ and F have the previously listed properties. Let

$$x_0 \colon M \hookrightarrow N$$

be the embedding of a hypersurface M_0, which is graphical over $\mathcal{S}_0$, i.e. there exists $u \in C^\infty(\mathcal{S}_0, (R_0,\infty))$ such that

$$M_0 = \{(u(y), y) \colon y \in \mathcal{S}_0\},$$

and such that all its n-tuples of principal curvatures belong to Γ.

(i) Assume that either of the following properties hold

(a) σ has non-negative sectional curvature.

(b) $F = n\dfrac{H_{k+1}}{H_k}, \quad 0 \leq k \leq n-1.$

Then there exists a unique immortal solution

$$x\colon [0,\infty)\times M \to N$$

of

$$\dot{x} = \frac{1}{F^p}\nu$$
$$x(0,\cdot) = x_0,$$

which is also graphical over $\mathcal{S}_0$*, i.e.* $\langle \nu, \partial_r\rangle > 0$*.*

(ii) Assume σ *has non-negative sectional curvature and each of the following properties:*

(*A*) *Assumption 2.3.1 holds.*

(*B*) $\sup_{r>0}\vartheta'(r) < \infty$ *and* $p = 1 \quad \Rightarrow \quad \widehat{\underline{\mathrm{Rc}}} > 0$ *and* $F = n\dfrac{H_{k+1}}{H_k}, \quad 0 \le k \le n-1.$

(*C*) $\sup_{r>0}\vartheta'(r) = \infty$ *and* $p = 1 \quad \Rightarrow \quad \liminf_{r\to\infty}\dfrac{\vartheta''\vartheta}{\vartheta'^2} > 0.$

Then the flow hypersurfaces become umbilical at the rate

$$\left|h^i_j - \frac{\vartheta'}{\vartheta}\delta^i_j\right| \le ct\frac{\vartheta'^{1-p(p+1)}}{\vartheta}, \tag{2.10}$$

where the t*-factor may be dropped in case* $p < 1$ *or bounded* ϑ' *and may even be replaced by* $e^{-\alpha t}$ *for some positive* α *if* ϑ' *is bounded and* $p = 1$*.*

A detailed discussion of the technical assumptions in this theorem, as well as an extensive bibliography of works on inverse curvature flows can be found in the introduction of Appendix A4. Let us only mention the most important ones here. As already described in Chapter 1, the inverse mean curvature in the Euclidean space, hyperbolic space and the sphere has been very useful to prove geometric inequalities, cf. [40, 45, 47, 53, 63, 103] and [81]. In more general warped products, among which are the anti-de Sitter-Schwarzschild manifolds and the Reissner-Nordström manifolds, an estimate like (2.10) was enough to obtain Minkowski and Penrose type inequalities. Hence the aim of the paper in Appendix A4 is to provide the estimate (2.10) in a setting as general as possible, with the hope that it will be possible to obtain interesting applications in the future.

The method of proof for Theorem 2.3.2 is straightforward by getting C^0, C^1 and curvature estimates via maximum principle and then to refine the asymptotics with suitable test functions. Finding such test functions can be tricky, but the search for them is surely inspired by previous works on this topic, although from time to time a refinement is necessary, especially since we also allow powers p of F, we do *not* impose a lower bound on the ambient sectional curvature and, more importantly, the fractions in Assumption 2.3.1 are not assumed to converge. This has, to the best of my knowledge, never been treated before, except in the previous work by J.S. on the inverse mean curvature flow [96]. Also note, that spherical ambient spaces, i.e. $\vartheta'' < 0$ are not included. In these cases there is no long-time existence and other methods are necessary, see [47] for an approach using dual flows and [81] for an approach using the geometry of spherically convex bodies.

One conceptual problem with the use of purely expanding curvature flows, such as the inverse mean curvature flow, in deducing geometric inequalities for compact hypersurfaces, is that the structure of the ends of the possibly unbounded ambient manifold are crucial for the type of convergence results one gets. However, depending on the geometric inequality for a hypersurface $M \subset N$ one would like to prove, it is reasonable to expect that only the local structure of N around M is relevant, at least when searching for critical points. Hence one strategy, for example to prove an Alexandrov-Fenchel inequality as in (1.2), could be to find a flow which leaves the right hand side constant and decreases the left hand side. Often this is not possible using a pure curvature flow, but by using a flow with constraining extra terms. The first such flow, the so-called *volume preserving mean curvature flow*, was studied by Huisken [61] for convex hypersurfaces and by Gage [35] for convex curves. It reads

$$\dot{x} = \left(\fint H - H \right) \nu. \tag{2.11}$$

This flow preserves the volume enclosed by the hypersurface and decreases its surface area. Hence it is naturally suited to prove the isoperimetric inequality. However, due to its non-local nature, it is analytically hard to study. Similar non-local flows in all kinds of variants (area preserving, quermassintegral preserving) have been studied in the Euclidean and the hyperbolic space, e.g. [5, 7, 17, 67, 75, 80, 85, 86, 87, 99]. Furthermore, there are perturbation and stability results, also including more general ambient spaces, [1, 30, 59]. One analytical drawback of (2.11) is, that due to the nonlocal nature, one can not prove the existence of barriers in dependence on initial barriers. In this sense, also for (2.11) the global structure of the ambient manifold comes into play. A very elegant way to overcome these obstructions was given by Pengfei Guan and Junfang Li in [54]. They constructed the flow

$$\dot{x} = (n\vartheta' - Hs)\nu, \tag{2.12}$$

where $s = \langle \vartheta \partial_r, \nu \rangle$ is the generalised support function. Due to the Minkowski identity

$$\int_M n\vartheta' = \int_M Hs,$$

this flow is volume preserving. It can also be shown that it is surface area decreasing. The big advantage is, that it solely depends on local quantities. Furthermore one can treat starshaped hypersurfaces, not only convex ones as in the non-local flows. Indeed, in [54], it is proved that (2.12) drives starshaped hypersurfaces in a simply connected spaceform exponentially to a geodesic sphere. In [56] the same result was accomplished in more general warped products, yielding the isoperimetric inequality for starshaped hypersurfaces in those warped products. It is tempting to define analogous flows which preserve other quantities, such as the higher order quermassintegrals. For instance, one can show that

$$\dot{x} = \left(\frac{n\vartheta'}{H} - s \right) \nu \tag{2.13}$$

is surface area preserving, while it decreases the total mean curvature. Hence, if one could prove convergence to a sphere in a warped product, one would obtain a Minkowski inequality for starshaped mean convex hypersurfaces. Unfortunately, the flow (2.13) is very tough to handle and, except for the Euclidean case, only unsatisfactory partial results could be obtained so far [12]. Some fully nonlinear versions of (2.12) in $\mathbb{R}^{n+1}$ were treated in [55]. Until today we were not able to prove convergence of solutions of (2.13). However, we were able to handle a related flow: Appendix A5, written jointly with Chao Xia, deals with long-time existence and convergence of solutions for the flow

$$\dot{x} = \left(\frac{n}{F} - \frac{s}{\vartheta'} \right) \nu. \tag{2.14}$$

It is not known to us whether this flow preserves any quantities, but it still has some nice monotonicity properties, which enabled us to prove some Minkowski type inequalities in warped products and a weighted isoperimetric type inequality in the hyperbolic space. The main theorems are the following:

2.3.3 Theorem. *Let $a, b \in \mathbb{R}$ and $(N, \bar{g})$ be the warped space $((a,b)\times\mathbb{S}^n, dr^2 + \vartheta^2(r)\sigma)$ with $\vartheta > 0$, $\vartheta' > 0$ and $\vartheta'' \geq 0$. Let F be the operator function associated to a symmetric, positive, strictly monotone, 1-homogeneous and concave curvature function $f \in C^\infty(\Gamma)$. Let x_0 be the embedding of a closed n-dimensional manifold M into N, such that $x_0(M)$ is a graph over the domain $\mathbb{S}^n$ and such that $\kappa \in \Gamma$ for all n-tuples of principal curvatures along $x_0(M)$. Then any solution x of (2.14) exists for all positive times and converges to a geodesic slice in the C^∞-topology.*

Furthermore we basically get the same result for the flow of strictly convex hypersurfaces of the sphere, cf. Thm. 1.1 in Appendix A5, if F is a quotient of elementary symmetric polynomials,

$$F = n\frac{H_k}{H_{k-1}}.$$

With the help of this result, we get the following geometric inequalities:

2.3.4 Theorem. *Let $N = (a,b)\times\mathbb{S}^n$ be equipped with one of the anti-de-Sitter Schwarzschild metrics or the hyperbolic metric, i.e.*

$$\vartheta' = \sqrt{1+\vartheta^2 - m\vartheta^{1-n}}, \quad m \geq 0.$$

Let $\Sigma \subset N$ be a closed, star-shaped and mean-convex hypersurface, given by the function $u\colon \mathbb{S}^n \to (a,b)$, and let

$$\Omega = \{(s,y) \in N \colon a \leq s \leq u(y), \quad y \in \mathbb{S}^n\}.$$

Then there hold

$$\int_\Sigma H\vartheta' d\mu - 2n\int_\Omega \frac{\vartheta'\vartheta''}{\vartheta} dN \geq \xi_1(|\Sigma|)$$

and

$$\int_\Sigma H\vartheta' d\mu - 2n\int_\Omega \frac{\vartheta'\vartheta''}{\vartheta} dN \geq \xi_0\left(\int_\Omega \vartheta' dN\right), \tag{2.15}$$

where ξ_0, ξ_1 are the associated monotonically increasing functions for radial coordinate slices. Equality holds if and only if Σ is a radial coordinate slice.

In particular, in the hyperbolic space, due to $\vartheta'' = \vartheta$, inequality (2.15) reduces to

$$\int_\Sigma H\vartheta' d\mu - (n+1)n\int_\Omega \vartheta' dN \geq n|\mathbb{S}^n|^{\frac{2}{n+1}}\left((n+1)\int_\Omega \vartheta' dN\right)^{\frac{n-1}{n+1}}, \tag{2.16}$$

where $\vartheta'(r) = \cosh u$. Equality in (2.16) holds if and only if Σ is a geodesic sphere centred at the origin. Chao Xia proved a Minkowski type inequality in [111] stating that for a closed horo-convex hypersurface $\Sigma \subset \mathbb{H}^{n+1}$ there holds

$$\left(\int_\Sigma \vartheta' d\mu\right)^2 \geq \frac{n+1}{n}\int_\Sigma H\vartheta' d\mu \int_\Omega \vartheta' dN.$$

Combining this with (2.16), we get:

2.3.5 Theorem. *Let Σ be a closed horo-convex hypersurface in $\mathbb{H}^{n+1}$ with the origin lying inside Ω. Then, with equality on origin-centred spheres,*

$$\int_\Sigma \vartheta' d\mu \geq \left[\left((n+1)\int_\Omega \vartheta' dN\right)^2 + |\mathbb{S}^n|^{\frac{2}{n+1}}\left((n+1)\int_\Omega \vartheta' dN\right)^{\frac{2n}{n+1}}\right]^{\frac{1}{2}}.$$

2.3.6 Remark. Theorem 2.3.5 already appeared in the paper [38], where it is the case $k=0$ in Thm. 9.2. However, their proof relies on an *invalid* inequality, namely [38, equ. (9.8)], which states

$$|\Sigma|^{\frac{n+1}{n}} \geq |\mathbb{S}^n|^{\frac{1}{n}}\int_\Sigma s\ d\mu.$$

This inequality is already incorrect on geodesic spheres not centred at the origin. Theorem 2.3.5 fixes this gap in the proof of [38, Thm. 9.2].

2.3.7 Remark. Please note, that the notation here deviates from the notation used in Appendix A5. When writing the paper, we compromised on a different notation, but in order to have a consistent notation within the summary chapter of this thesis, I decided to change it back for the current section.

It is in order to say a few words about the proof. The rough strategy is straightforward, obtaining barriers, gradient and curvature estimates. However, due to the additional terms s and ϑ' in the flow speed, the computations as well as the balancing of all the new (compared to a pure curvature flow) terms are delicate. The final round limiting shape can be deduced from the existence of some monotone quantities, which allow to get the spherical shape with the help of some rigidity results, e.g. Brendle's Heintze-Karcher type inequality [11]. On several occasions we proved some estimates, such as for the gradient or the curvature function F, in a greater generality than needed, with the hope that they might be useful later.

The main technical reason, why the flow (2.14) is easier to handle than (2.13) is that in the latter the lower order term ϑ' is coupled with the curvature. This feature gives some mixed derivative terms of ∇H and ∇u, which we were not able to deal with. It is a long-term goal of J.S. to overcome these difficulties.

To finish this section it is still left to precisely describe the contribution of J.S. to the paper "Locally constrained inverse curvature flows" in Appendix A5.

The introductory **sections 1-3** of this paper consist of the statement of the results, notational conventions and evolution equations. They are **written approximately to an equal amount by both authors.**

Sections 4 and 5 consist of the general gradient and F-bounds. The idea for the proof, i.e. the construction of test functions, as well as most of the writing, is **due to Chao Xia**.

Sections 6-8, which contain the preservation of convexity in sphere as well as estimates on the principal curvatures, were **elaborated and written by J.S.**, except for Prop. 7.4 (lower F-bound), the idea for which was given to J.S. by Chao Xia.

Section 9 about the geometric inequalities is due to Chao Xia.

2.4 Appendices A6 and A7

Until now, all curvature flows we considered are evolving closed hypersurfaces in a given ambient space. In contrast, the final Appendices A6 and A7, jointly written with Ben Lambert, deal with the IMCF

$$\dot{x} = \frac{1}{H}\nu \tag{2.17}$$

of hypersurfaces $M_t \subset \mathbb{R}^{n+1}$ with boundary. Thus, in order to obtain uniqueness of a solution, we have to impose a boundary condition. We chose perpendicularity to the unit sphere from the inside. In Appendix A6 we prove a convergence result for the IMCF of strictly convex hypersurfaces satisfying this perpendicularity condition, cf. Theorem 2.4.1 below. Free boundary hypersurfaces with prescribed contact angle have become a classical topic in geometric analysis during the past decades, for example as capillary or minimal surfaces, e.g. [10, 78, 88, 91, 92, 105] or in the context of Steklov eigenvalues and the Dirichlet-to-Neumann problem [31, 32, 33].

In the context of curvature flows there has also been a wide interest in flows that satisfy Dirichlet or Neumann conditions, such as the mean curvature flow of graphs [2, 62], the mean curvature flow with perpendicular free boundary condition [16, 29, 49, 69, 70, 71, 79, 100, 101] and the inverse mean curvature flow with perpendicular Neumann condition [83, 84].

The following theorem is the main result of Appendix A6, where $\tilde{\nu}$ denotes the outward unit normal of the unit sphere $\mathbb{S}^n$ and $\mathbb{D}$ the n-dimensional unit disk.

2.4.1 Theorem. *Let*

$$x_0 \colon \mathbb{D} \hookrightarrow M_0 \subset \mathbb{R}^{n+1}$$

be the embedding of a smooth and strictly convex hypersurface with normal vector field ν_0, such that

$$\begin{aligned} x_0(\partial\mathbb{D}) &\subset \mathbb{S}^n, \\ \langle \dot{\gamma}(0), \tilde{\nu} \rangle &\geq 0 \quad \forall \gamma \in C^1((-\epsilon, 0], M_0) \colon \gamma(0) \in \partial x_0(\mathbb{D}), \\ \left\langle \nu_{0|\partial\mathbb{D}}, \tilde{\nu}_{|\partial\mathbb{D}} \right\rangle &= 0. \end{aligned} \tag{2.18}$$

Then there exists a finite time $T^* < \infty$, $\alpha > 0$ and a unique solution

$$x \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T^*) \times \mathbb{D}, \mathbb{R}^{n+1}) \cap C^\infty((0, T^*) \times \mathbb{D}, \mathbb{R}^{n+1})$$

of (2.17) with initial hypersurface M_0, such that the flow hypersurfaces M_t share the properties in (2.18) and the embeddings x_t converge to the embedding of a flat unit disk as $t \to T^*$, in the sense that the height of the $M_t = x(t, \mathbb{D})$ over this disk converges to 0.

2.4.2 Remark. Please note, that the notation used in Theorem 2.4.1 deviates from the one used in Appendix A6 in order to be consistent with the notation used in the rest of Chapter 2.

To the best of our knowledge, for the inverse mean curvature flow with boundary the only previous convergence result is due to Marquardt [83] who showed, in the spirit of Gerhardt's seminal work [40], that a hypersurface with boundary, graphical over some portion of the sphere $\mathbb{S}^n$ and perpendicular to the cone in $\mathbb{R}^{n+1}$ over this portion, exists for all time, is driven to infinity by IMCF and after rescaling to unit surface area converges to the corresponding portion of a sphere.

The big qualitative difference between Marquardt's and our result Theorem 2.4.1 is that the geometry of the supporting hypersurface $\mathbb{S}^n$ forces the mean curvature H to drop to zero even quicker, resulting in a finite time singularity formation: The smooth flow, starting from strictly convex initial data, can only exist for a finite time and tends to the unique disk-type minimal surface perpendicular to $\mathbb{S}^n$, namely the flat unit disk. The norm of convergence is at least $C^{1,\alpha}$. We are not aware whether the norm of convergence can be improved. This remains an interesting open question. Our result should be compared to a very recent preprint by Daskalopoulos and Huisken [23], who consider the IMCF of entire graphs over $\mathbb{R}^n$ with a prescribed growth at infinity. They also obtain maximal existence on a finite interval, with $C^{1,\alpha}$ convergence to the flat plane.

The proof of Theorem 2.4.1 combines analytical tools with methods from spherical convex geometry. Let us list the main steps of the proof.

(i) A hypersurface, which has the properties (2.18), can be written as a graph in a conformally flat coordinate system, built from rotating a special class of Möbius transformations of the 2-plane. Hence the problem (2.17), (2.18) can be written as a scalar fully nonlinear PDE with a classical Neumann boundary condition. Short-time existence then follows from standard parabolic theory.

(ii) Any strictly convex hypersurface $M_t \subset \mathbb{R}^{n+1}$ with boundary satisfying (2.18) induces a strictly convex hypersurface $\partial M_t \subset \mathbb{S}^n$, which lies in the interior of a hemisphere $\mathcal{H}(e_0)$ around some $e_0 \in \mathbb{S}^n$ due to a classical result by Do Carmo and Warner [28]. We obtain height estimates for M_t over the plane $e_0^\perp$, in the sense that as long as ∂M_t is strictly separated from the equator $e_0^\perp \cap \mathbb{S}^n$, we have

$$\langle x(t,\cdot), e_0\rangle \geq c_0 > 0. \tag{2.19}$$

(iii) Using a suitably built barrier function, we use the maximum principle to show that given a height estimate like (2.19), we get a lower bound on the mean curvature, as well as preserved convexity. Using the short-time existence result, we obtain that the boundaries ∂M_t can not remain strictly in an open hemisphere up to the maximal time of existence T^*.

(iv) The boundaries ∂M_t bound strictly spherically convex bodies $\hat{M}_t \subset \mathbb{S}^n$, which are increasing as sets. Hence there exists a limiting body $\hat{M}_{T^*}$, for which we can prove two crucial properties:

   (a) It satisfies a uniform interior ball condition, i.e. there exists $R > 0$, such that for all $p \in \partial\hat{M}_{T^*}$ there exists an open ball $B_R(x_p) \subset \operatorname{int}(\hat{M}_{T^*})$ with

$$\partial B_R(x_p) \cap \partial\hat{M}_{T^*} = \{p\}$$

   and

   (b) $\hat{M}_{T^*}$ is a *weakly convex body in the hemisphere* $\mathcal{H}(e_0)$, meaning that for all $p, q \in \hat{M}_{T^*}$ there exists a minimizing geodesic γ connecting p, q and contained in $\hat{M}_{T^*}$.

   For such sets $\hat{M}_{T^*} \subset \mathbb{S}^n$, Matthias Makowski and J.S. have proved in [81]:

   $\hat{M}_{T^*}$ is either strictly contained in an open hemisphere or is equal to a closed hemisphere.

   The first option is ruled out since otherwise we could extend the flow beyond T^*. Hence $\partial\hat{M}_{T^*}$ is equal to an equator and we obtain uniform convergence of ∂M_t to an equator of $\mathbb{S}^n$, as $t \to T^*$.

(v) The height estimates imply that the M_t converge uniformly to the corresponding flat unit disk and the $C^{1,\alpha}$-convergence follows due to uniform curvature estimates, which we get from a simple maximum principle argument.

The **sections 1-3** of Appendix A6, introduction, notation and evolution equations, **were written by both Ben Lambert and J.S. to an equal amount**.

The **height estimates** in section 4 in this rigorous form are **due to J.S.**, where however, some intuition for the geometric situation was achieved through conversations with Ben Lambert.

Ben Lambert came up with the Möbius coordinates in section 5, which allowed us to reduce the flow to a scalar evolution problem giving short-time existence.

Sections 6 and 7, especially the construction of the barrier for $1/H$, the application of the rigidity result from [81] and the completion of the proof, are **due to J.S.**

The counterexample in the appendix of the paper, saying that such a result is not possible if we replace the supporting sphere by an ellipsoid, **is due to Ben Lambert**.

In Appendix A7 we apply Theorem 2.4.1 to prove a geometric inequality for convex free boundary hypersurfaces in the unit ball which meet the sphere orthogonally:

2.4.3 Theorem. *Let $n \geq 3$ and $M^n \subset \mathbb{R}^{n+1}$ be a smoothly embedded n-disk, such that M^n is a convex hypersurface perpendicular to $\mathbb{S}^n$ from the inside, i.e. it satisfies (2.18). Then there holds*

$$\frac{1}{2}|M|^{\frac{2-n}{n}}\int_M H^2 + \omega_n^{\frac{2-n}{n}}|\partial M| \geq \omega_n^{\frac{2-n}{n}}|\mathbb{S}^{n-1}| \tag{2.20}$$

and equality holds if and only if M is a perpendicularly intersecting hyperplane.

The case $n = 2$ was treated by Volkmann [106]. Using other methods, he even obtained a better estimate for a larger class of hypersurfaces, replacing the $1/2$ factor by $1/4$, which yields a conformally invariant quantity on the left hand side and characterises the equality case by perpendicularly intersecting spheres. However, his methods are restricted to the case $n = 2$. The proof of Theorem 2.4.3 is not quite as straightforward as it is usually the case, once one has a converging curvature flow and a monotone quantity (the left hand side of (2.20), call it Q, is non-increasing.) Of course Q is monotone along the flow we consider in Theorem 2.4.1, the IMCF perpendicular to the sphere, but the convergence result only holds for strictly convex initial data. Hence not all hypersurfaces M, for which Theorem 2.4.3 claims an estimate, are admissable for the flow. This is in big contrast to previous works using the flow method to prove geometric inequalities, such as the ones mentioned in Chapter 1. Let us have a look at the main steps of the proof, which also describe how to deal with weakly convex hypersurfaces and how we treat the limiting case (usually the limiting case is settled by observing that the monotone quantity is constant on the limiting object, but in our case for the *minimal* limiting object the flow is not defined.)

(i) Prove (2.20) in case that M is strictly convex. This follows by a standard argument using the monotonicity of Q along (2.17).

(ii) Approximate a general convex hypersurface M in C^2 by strictly convex hypersurfaces $\tilde{M}$ satisfying (2.18), in case that M is not a flat disk (in which case we would be done anyway). If M is convex and not a flat disk, we prove that M contains a strictly convex region and hence the *mean curvature flow with boundary* with initial data M has strictly convex flow leaves $\tilde{M}_\tau$, $\tau > 0$, immediately. They converge backwards to M as $\tau \to 0$ in the norm of $C^{2,\alpha}$. Hence we obtain (2.20) in the general case.

(iii) For the equality case we use a contradiction argument. The idea is as follows: Suppose we have equality in (2.20) and M is not a flat disk. In Lemma 3.4 we obtain a volume estimate for M,

$$|M| \leq \omega_n - c_{\partial M},$$

where ω_n is the volume of the unit n-ball and $c_{\partial M}$ only depends on the spherical outer radius of $\partial M \subset \mathbb{S}^n$. The monotonicity calculation in the proof of Lemma 2.3, especially equ. (2.25), yields that the rate of diminishment of Q is strictly bounded away from zero, as long as the volume is bounded away from ω_n. Hence, approximating M in C^2 by strictly convex hypersurfaces M_ϵ and starting IMCF from each M_ϵ (with uniformly positive existence time T due to the exponential volume growth of IMCF), we see that $Q_\epsilon(t)$ drops at a strictly negative rate for at least time T. Since

$$Q_\epsilon(0) \to \omega_n^{\frac{2-n}{n}} |\mathbb{S}^{n-1}|, \quad \epsilon \to 0,$$

given a small $\epsilon > 0$, $Q_\epsilon(t)$ will at some $t > 0$ eventually drop below $\omega_n^{\frac{2-n}{n}} |\mathbb{S}^{n-1}|$, which contradicts the inequality in the strictly convex case.

In Appendix A7 the work was distributed among Ben Lambert and J.S. in the following way.

Section 2, the monotonicity calculation in the strictly convex case, was **performed together** while Ben Lambert was on a research visit in Freiburg in summer 2016. This was the starting point of the paper.

The strategy to obtain the weakly convex case as well as the limiting case, especially the idea described in item (iii), **is due to J.S.**

The great contribution of Ben Lambert was to provide the mean curvature flow argument from Thm. 3.2 and Cor. 3.3 to approximate the weakly convex hypersurface M.

The existence of a strictly convex point on the weakly convex hypersurface M in **Lemma 3.1, as well as** the volume estimate in **Lemma 3.4 are due to J.S.**

Finishing up the proof of Theorem 2.4.3 in **section 4 was** then straightforward and **provided by both authors to an equal amount.**

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Chapter 3

APPENDIX

Appendix A1

ISOTROPIC FUNCTIONS REVISITED

by Julian Scheuer
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ISOTROPIC FUNCTIONS REVISITED

JULIAN SCHEUER

Abstract. To a real n-dimensional vector space V and a smooth, symmetric function f defined on the n-dimensional Euclidean space we assign an associated operator function F defined on linear transformations of V. F shall have the property that, for each inner product g on V, its restriction F_g to the subspace of g-selfadjoint operators is the isotropic function associated to f. This means that it acts on these operators via f acting on their eigenvalues. We generalize some well known relations between the derivatives of f and each F_g to relations between f and F, while also providing new elementary proofs of the known results. By means of an example we show that well known regularity properties of F_g do not carry over to F.

1. Introduction

Consider a function $f \in C^\infty(\mathbb{R}^n)$ which is *symmetric*, i.e.

$$f(\kappa_1, \dots, \kappa_n) = f(\kappa_{\pi(1)}, \dots, \kappa_{\pi(n)}) \quad \forall \pi \in \mathcal{P}_n,$$

where $\mathcal{P}_n$ is the permutation group on n elements. Let V be a real, n-dimensional vector space and $\mathcal{L}(V)$ be the vector space of linear operators on V. If V carries an inner product g, on the vector subspace $\Sigma_g(V) \subset \mathcal{L}(V)$ of g-selfadjoint operators one can define a map

$$\begin{aligned} F_g \colon \Sigma_g(V) &\to \mathbb{R} \\ A &\mapsto f(\mathrm{EV}(A)), \end{aligned}$$

where $\mathrm{EV}(A) = (\kappa_1, \dots, \kappa_n)$ denotes the ordered n-tuple of real eigenvalues of A. In [2] J. Ball proved that if $f \in C^r(\mathbb{R}^n)$, $r = 1, 2, \infty$, the function F_g is also of class C^r. Furthermore, using Schauder theory, he showed that if $f \in C^{r,\alpha}(\mathbb{R}^n)$, $r \in \mathbb{N}$, $0 < \alpha < 1$, then also F is in the respective function class. Also compare [11, Sec. 2.1] for a detailed proof of these regularity results. For $r \geq 3$, the implication

$$f \in C^r(\mathbb{R}^n) \quad \Rightarrow \quad F_g \in C^r(\Sigma_g(V))$$

was proven in [19].

In these results one always starts with an inner product space (V, g). In many applications one has to deal with a whole family of such spaces, where g may vary. For example in geometric curvature problems one is often faced with a map F being evaluated on the Weingarten tensor $\mathcal{W}$, an endomorphism field with values in the tensor bundle of linear transformations of the tangent spaces. From point to point, these linear maps $\mathcal{W}(x)$ are self-adjoint with respect to different metrics, so one has to be careful with the domain of F.

One may observe, that for the most natural symmetric functions, e.g.

$$s_1 = \sum_{i=1}^n \kappa_i \quad \text{or} \quad s_n = \prod_{i=1}^n \kappa_i$$

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there is no ambiguity about how to define F even on the whole space $\mathcal{L}(V)$ and not only on some $\Sigma_g(V)$. Namely for s_1 just set

$$F(A) = S_1(A) = \operatorname{tr}(A)$$

and for s_n set

$$F(A) = S_n(A) = \det(A).$$

The functions s_1 and s_n are special cases of the *elementary symmetric polynomials* s_k, $1 \leq k \leq n$, cf. Definition 2.1, to which we associate

$$S_k(A) = \frac{1}{k!}\frac{d^k}{dt^k}\det(I + tA)_{|t=0}.$$

It is true that every symmetric function $f \in C^\infty(\Gamma)$ on a symmetric open set $\Gamma \subset \mathbb{R}^n$ can be written as a function of the s_i,

$$f = \rho(s_1, \dots, s_n),$$

where $\rho \in C^\infty(U)$ for some open $U \subset \mathbb{R}^n$, cf. [12]. In case $f \in C^r(\Gamma)$, ρ will in general have less regularity, cf. [3]. In both cases the function

$$F = \rho(S_1, \dots, S_n)$$

is defined on an open set $\Omega \subset \mathcal{L}(V)$ and satisfies

$$F(A) = f(\mathrm{EV}(A))$$

for all $\mathbb{R}$-diagonalisable $A \in \mathcal{L}(V)$ with eigenvalues in Γ. Hence F can be differentiated in all directions of $\mathcal{L}(V)$.

The aim of this short note is a transfer of some well known and often used relations between derivatives of F and f to the new situation, that F can be differentiated in all of $\mathcal{L}(V)$. In previous treatments of this, only the relation between f and F_g was studied for some fixed metric g, compare for example [1, 2, 4, 8, 9, 11, 13, 16, 18, 19]. Our approach is by direct calculation of the proposed relations for the elementary symmetric polynomials and then to transfer them to general functions. Note that this approach also provides a new, quite elementary proof of the corresponding results for the pair (f, F_g) with fixed inner product g, as obtained in [1, Thm. 5.1] and [11, Lemma 2.1.14].

The motivation to write this note came up during the preparation of [7], where we had to apply derivatives of F_g to some non-g-selfadjoint operators, so the need for a globally defined F was apparent. For illustration, have a look at the following simple example:

1.1. *Example.* Let f be the second power sum,

$$f(\kappa) = \sum_{i=1}^{n} \kappa_i^2, \quad F(A) = \operatorname{tr}(A^2),$$

then F is clearly the associated operator function for f and F is defined on whole $\mathcal{L}(V)$. f is a convex function of the κ_i. However,

$$F\colon \mathcal{L}(V) \to \mathbb{R}$$

is *not* convex: Indeed there holds

$$dF(A)B = 2\operatorname{tr}(A \circ B),$$

$$d^2F(A)(B, C) = 2\operatorname{tr}(B \circ C)$$

and hence

$$d^2F(A)(\eta, \eta) = 2\operatorname{tr}(\eta^2) < 0$$

for a nonzero skew-symmetric (with respect to a basis of eigenvectors of A) η.

The fact that F is in general not convex, when considered as a function on $\mathcal{L}(V)$, caused trouble in the preparation of [7], where we had to estimate the term $d^2F(\dot{\mathcal{W}}, \dot{\mathcal{W}})$ along some curvature flow. Here $\dot{\mathcal{W}}$ is the evolution of the Weingarten tensor. For the particular flow considered in [7], we could not prove the symmetry of $\dot{\mathcal{W}}$. This trouble was the main motivation to write this note and to extend the formulas for derivatives of F, as in Proposition 2.8.

2. Symmetric functions and associated operator functions

For an n-dimensional, real vector space V, the aim of this section is to deduce relations between the derivatives of the functions f and F as described in the introduction. First we fix some definitions and notation.

2.1. **Definition.** On $\mathbb{R}^n$ we denote the elementary symmetric polynomials for $1 \leq k \leq n$ by s_k,

$$s_k(\kappa_1, \ldots, \kappa_n) := \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} \kappa_{i_j}$$

and the power sums for all $k \in \mathbb{N}$ by p_k,

$$p_k(\kappa) = \sum_{i=1}^{n} \kappa_i^k.$$

2.2. **Definition.** (i) Let V be an n-dimensional real vector space and $\mathcal{D}(V) \subset \mathcal{L}(V)$ be the set of diagonalisable endomorphisms. Then we denote by EV the eigenvalue map, i.e.

$$\begin{aligned} \mathrm{EV}\colon \mathcal{D}(V) &\to \mathbb{R}^n/\mathcal{P}_n \\ A &\mapsto (\kappa_1, \ldots, \kappa_n), \end{aligned}$$

where $\kappa_1, \ldots, \kappa_n$ denote the eigenvalues of A and $\mathcal{P}_n$ is the permutation group of n elements.

(ii) Let $\Gamma \subset \mathbb{R}^n$ be open and symmetric, then we define

$$\mathcal{D}_\Gamma(V) = \mathrm{EV}^{-1}(\Gamma/\mathcal{P}_n).$$

2.3. *Remark.* Note that EV is continuous, compare [21].

2.4. **Lemma.** *Let V be an n-dimensional real vector space. Then for all $k \in \mathbb{N}$ there exists a function $P_k \in C^\infty(\mathcal{L}(V))$ with*

$$P_k(A) = p_k \circ \mathrm{EV}(A) \quad \forall A \in \mathcal{D}(V).$$

Proof. Simply set

$$P_k(A) = \mathrm{tr}(A^k).$$

Then there holds

$$P_k(A) = p_k(\mathrm{EV}(A)) \quad \forall A \in \mathcal{D}(V).$$

$\square$

Since the P_k are smooth, we want to investigate the structure of their derivatives.

2.5. **Proposition.** *Let V be an n-dimensional real vector space. Let $U \subset \mathbb{R}^m$ be open and $\psi \in C^r(U)$, $r \geq 1$. Then the function*

$$f = \psi(p_1, \dots, p_m)$$

is defined on an open symmetric set $\Gamma \subset \mathbb{R}^n$ and the function $F = \psi(P_1, \dots, P_m)$ is defined on an open set $\Omega \subset \mathcal{L}(V)$. There holds

$$F_{|\mathcal{D}_\Gamma(V)} = f \circ \mathrm{EV}_{|\mathcal{D}_\Gamma(V)}$$

and the derivatives of F evaluated at a fixed $A \in \Omega$ are given by

$$dF(A)B = \operatorname{tr}(F'(A) \circ B) = \sum_{l=1}^{m} l \frac{\partial \psi}{\partial P_l} \operatorname{tr}\left(A^{l-1} \circ B\right) \quad \forall B \in \mathcal{L}(V), \tag{2.1}$$

where

$$F'(A) = \sum_{l=1}^{m} l \frac{\partial \psi}{\partial P_l} A^{l-1}. \tag{2.2}$$

Proof. Only the formula for dF has to be checked, while all other statements are obvious. The function $P_1(A) = \operatorname{tr}(A)$ is linear and hence

$$dP_1(A)B = \operatorname{tr}(B) \quad \forall A, B \in \mathcal{L}(V).$$

Furthermore by the chain rule there holds

$$dP_k(A)B = d(P_1(A^k))(A)B = k \operatorname{tr}(A^{k-1} \circ B) \quad \forall A, B \in \mathcal{L}(V). \tag{2.3}$$

Thus

$$dF(A)B = \sum_{l=1}^{m} \frac{\partial \psi}{\partial P_l} dP_l(A)B = \operatorname{tr}(F'(A) \circ B) \tag{2.4}$$

and hence the proof is complete. $\square$

2.6. *Remark.* It is well known that the elementary symmetric polynomials s_k are functions of the p_k, cf. [17], and hence Proposition 2.5 also applies to these.

2.7. **Corollary.** *Let V be an n-dimensional real vector space and let f and F be as in Proposition 2.5. Suppose $A \in \mathcal{D}_\Gamma(V)$. Then the endomorphisms $F'(A)$ and A are simultaneously diagonalisable. For a basis $(e_1, \dots, e_n)$ of eigenvectors for A with eigenvalues $\kappa = (\kappa_1, \dots, \kappa_n)$, the eigenvalue F^i of $F'(A)$ with eigenvector e_i is given by*

$$F^i(A) = \frac{\partial f}{\partial \kappa_i}(\kappa). \tag{2.5}$$

Proof. That $F'(A)$ and A are simultaneously diagonalisable follows from (2.2) immediately. Let (κ_i) be the eigenvalues of A. The eigenvalues of F' can be read off (2.2). They are

$$F^i = \sum_{l=1}^{m} l \frac{\partial \psi}{\partial p_l} \kappa_i^{l-1} = \frac{\partial f}{\partial \kappa_i},$$

due to the chain rule. $\square$

There also follows a representation for the second derivatives of the function F. Proofs for the case that F is defined on the subspace of selfadjoint operators with respect to a fixed metric can be found in [1, Thm. 5.1], [11, Lemma 2.1.14] and [19], where in the latter even higher derivatives are treated. The proof presented here is by direct differentiation of (2.4). It extends similar proofs used in the context of tensor valued functions in [4, 5, 8, 9] to the case $n > 3$ and diagonalisable A. There are several other very recent results [15], which address similar questions in the context of operator monotone functions and k-isotropic functions. Also compare the comprehensive thesis [14], as well as [16] and [18].

2.8. **Proposition.** *Let V be an n-dimensional real vector space and let F and f be as in Proposition 2.5 with $r \geq 2$. Let $A \in \mathcal{D}_\Gamma(V)$ and let (η^i_j) be a matrix representation of some $\eta \in \mathcal{L}(V)$ with respect to a basis of eigenvectors of A. Then there holds*

$$d^2F(A)(\eta,\eta) = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial\kappa_i\partial\kappa_j}\eta^i_i\eta^j_j + \sum_{i\neq j}^n \frac{\frac{\partial f}{\partial\kappa_i} - \frac{\partial f}{\partial\kappa_j}}{\kappa_i - \kappa_j}\eta^i_j\eta^j_i, \tag{2.6}$$

where f is evaluated at the n-tuple (κ_i) of corresponding eigenvalues. The latter quotient is also well defined in case $\kappa_i = \kappa_j$ for some $i \neq j$.

Proof. Starting from (2.4) we can calculate for all $A \in \Omega \subset \mathcal{L}(V)$ and $B, C \in \mathcal{L}(V)$, that

$$\begin{aligned} d^2F(A)(B,C) &= \sum_{k,l=1}^m \frac{\partial^2\psi}{\partial P_l\partial P_k}(dP_l(A)B)(dP_k(A)C) \\ &\quad + \sum_{k=1}^m \frac{\partial\psi}{\partial P_k} d^2P_k(A)(B,C). \end{aligned} \tag{2.7}$$

From (2.3) we obtain, already inserting $B = C = \eta = \hat\eta + \tilde\eta$, where $\hat\eta$ is the diagonal part of η in a basis of eigenvectors for A and $\tilde\eta$ is the corresponding off-diagonal part $\tilde\eta = \eta - \hat\eta$,

$$\begin{aligned} d^2P_k(A)(\eta,\eta) &= k\sum_{l=1}^{k-1} \operatorname{tr}(A^{l-1}\circ\eta\circ A^{k-1-l}\circ\eta) \\ &= k\sum_{l=1}^{k-1} \big(\operatorname{tr}(A^{l-1}\circ\hat\eta\circ A^{k-1-l}\circ\hat\eta) \\ &\qquad + \operatorname{tr}(A^{l-1}\circ\tilde\eta\circ A^{k-1-l}\circ\tilde\eta)\big). \end{aligned} \tag{2.8}$$

Using the specific basis of eigenvectors we get

$$\begin{aligned} d^2P_k(A)(\eta,\eta) &= k(k-1)\sum_{i=1}^n \kappa_i^{k-2}(\eta^i_i)^2 + k\sum_{l=1}^{k-1}\sum_{i,j=1}^n \kappa_i^{l-1}\kappa_j^{k-1-l}\tilde\eta^i_j\tilde\eta^j_i \\ &= \sum_{i,j=1}^n \frac{\partial^2 p_k}{\partial\kappa_i\partial\kappa_j}\eta^i_i\eta^j_j + \sum_{i\neq j} k\frac{\kappa_i^{k-1}-\kappa_j^{k-1}}{\kappa_i-\kappa_j}\eta^i_j\eta^j_i \\ &= \sum_{i,j=1}^n \frac{\partial^2 p_k}{\partial\kappa_i\partial\kappa_j}\eta^i_i\eta^j_j + \sum_{i\neq j} \frac{\frac{\partial p_k}{\partial\kappa_i}-\frac{\partial p_k}{\partial\kappa_j}}{\kappa_i-\kappa_j}\eta^i_j\eta^j_i. \end{aligned} \tag{2.9}$$

Hence the claimed result holds for the power sums. Returning to (2.7) we obtain, also using Corollary 2.7,

$$\begin{aligned} d^2F(A)(\eta,\eta) &= \sum_{k,l=1}^{m} \frac{\partial^2\psi}{\partial P_l \partial P_k}(dP_l(A)\hat{\eta})(dP_k(A)\hat{\eta}) + \sum_{k=1}^{m}\frac{\partial\psi}{\partial P_k} d^2P_k(A)(\eta,\eta) \\ &= \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial\kappa_i\partial\kappa_j}\eta_i^i\eta_j^j + \sum_{k=1}^{m}\frac{\partial\psi}{\partial P_k}\sum_{i\neq j}\frac{\frac{\partial p_k}{\partial\kappa_i}-\frac{\partial p_k}{\partial\kappa_j}}{\kappa_i-\kappa_j}\eta_j^i\eta_i^j, \end{aligned}$$

from which the claim follows due to the chain rule. Also in this formula, the quotient makes sense even if $\kappa_i = \kappa_j$, since the singularity in this fraction is removable, as can be seen from (2.9). $\square$

2.9. *Remark.* The representation formulae (2.5) and (2.6) are only valid a diagonalisable A, since their expressions make use of a particular basis of eigenvectors. Formulae which are valid for arbitrary $A \in \Omega$ are given, though a little less explicit, in (2.1) and (2.8). They are still easy enough to serve as a computational tool, particularly in low dimensions.

Although in the previous proof we have already seen an explicit expression for the quotient term in (2.6), we want to at least mention another representation. It appeared in [11, Lemma 2.1.14] and [19], also compare [10, Lemma 2]. The proof is similar to these references.

2.10. **Lemma.** *Let f be as in Proposition 2.5 with $r \geq 2$ and suppose that Γ is convex. Then there holds*

$$\frac{\frac{\partial f}{\partial\kappa_i}-\frac{\partial f}{\partial\kappa_j}}{\kappa_i-\kappa_j} = \frac{1}{2}\int_0^1 \left(\frac{\partial^2 f}{\partial\kappa_i^2} - 2\frac{\partial^2 f}{\partial\kappa_i\partial\kappa_j} + \frac{\partial^2 f}{\partial\kappa_j^2}\right),$$

where the integrand is evaluated along the line segment

$$\sigma(t) = \kappa + t\frac{\kappa_j-\kappa_i}{2}(e_i - e_j).$$

An alternative proof. Let us have a look at a second nice proof of Proposition 2.8, the idea of which appeared in [19, Lemma 3.2]. I owe thanks to the anonymous referee for the observation that this method can also be applied in our situation. It is based on the fact that the function F, as given in Proposition 2.5, is $\mathrm{Gl_n}(V)$-invariant:

$$F(SAS^{-1}) = F(A) \quad \forall A \in \mathcal{L}(V)\ \forall S \in \mathrm{Gl_n}(V). \tag{2.10}$$

In [19, Lemma 3.2] this property held for all orthogonal transformations S of a subspace of self-adjoint operators, but the proof basically carries over. Let us repeat it quickly here.

We suppose that all eigenvalues of A are mutually different. The general case can then be treated by approximation as in [19]. Differentiating the relation (2.10) with respect to A in direction of an arbitrary $\eta \in \mathcal{L}(V)$ we obtain for all $S \in \mathrm{Gl_n}(V)$, that

$$dF(SAS^{-1})(S\eta S^{-1}) = dF(A)(\eta). \tag{2.11}$$

In particular, choosing $S = e^{tW}$ for arbitrary $W \in \mathcal{L}(V)$, $t \in \mathbb{R}$, and differentiating (2.11) with respect to t at $t = 0$ gives

$$d^2F(A)(WA - AW, \eta) = dF(A)(\eta W - W\eta).$$

On the other hand, writing

$$\eta = \hat{\eta} + \tilde{\eta},$$

with diagonal $\hat\eta$ and off-diagonal $\tilde\eta$, we have

$$d^2F(A)(\eta,\eta) = d^2F(A)(\hat\eta,\hat\eta) + 2d^2F(A)(\hat\eta,\tilde\eta) + d^2F(A)(\tilde\eta,\tilde\eta).$$

With respect to a basis of eigenvectors for A and $F'(A)$ we define

$$W^i_j = \frac{\tilde\eta^i_j}{\kappa_j - \kappa_i},$$

which implies

$$W^i_k A^k_j - A^i_k W^k_j = \tilde\eta^i_j$$

and hence

$$d^2F(A)(\hat\eta,\tilde\eta) = dF(A)(\hat\eta W - W\hat\eta) = 0$$

and

$$d^2F(A)(\tilde\eta,\tilde\eta) = DF(A)(\tilde\eta W - W\tilde\eta) = \sum_{i\neq j} \frac{\frac{\partial f}{\partial\kappa_i} - \frac{\partial f}{\partial\kappa_j}}{\kappa_i - \kappa_j}\eta^i_j\eta^j_i.$$

Finally, since A and $\hat\eta$ are simultaneously diagonal, we have

$$\begin{aligned} d^2F(A)(\hat\eta,\hat\eta) &= \frac{d}{dt}\left(dF(A+t\hat\eta)(\hat\eta)\right)_{|t=0} \\ &= \frac{d}{dt}\left(\frac{\partial f}{\partial\kappa_i}(\kappa + t(\eta^i_i))\eta^i_i\right)_{|t=0} \\ &= \frac{\partial^2 f}{\partial\kappa_i\partial\kappa_j}(\kappa)\eta^i_i\eta^j_j \end{aligned}$$

and Proposition 2.8 follows. □

There is a slight advantage of the first proof of Proposition 2.8, namely that the calculation in (2.9) gives a precise description of why the term involving $\kappa_i - \kappa_j$ in the denominator also makes sense in case of coalescing eigenvalues.

3. Functions on bilinear forms

There is a useful relation of our maps $F\colon \Omega \subset \mathcal{L}(V) \to \mathbb{R}$ to maps which are defined on bilinear forms. First we need several definitions.

3.1. **Definition.** Let V be a finite dimensional real vector space.

(i) We denote the vector space of bilinear forms on V by $\mathcal{B}(V)$. The space of bilinear forms on the dual space V^* is denoted by $\mathcal{B}^*(V)$. The respective subsets of symmetric and positive definite forms will be denoted by $\mathcal{B}_+(V)$ and $\mathcal{B}^*_+(V)$.

(ii) For $a \in \mathcal{B}(V)$ and $b \in \mathcal{B}^*(V)$ we set

$$\begin{aligned} a_*\colon V &\to V^* \\ v &\mapsto a(v,\cdot) \end{aligned}$$

and

$$\begin{aligned} b^*\colon V^* &\to V \\ \phi &\mapsto J^{-1}\left(b(\phi,\cdot)\right), \end{aligned}$$

where $J\colon V \to V^{**}$ is the canonical identification given by

$$v \mapsto (\phi \mapsto \phi(v)).$$

(iii) Let $a \in \mathcal{B}(V)$ and $b \in \mathcal{B}^*(V)$, then we define $b * a \in \mathcal{L}(V)$ by contraction, i.e.

$$b * a = b^* \circ a_*.$$

(iv) For $g \in \mathcal{B}_+(V)$ we define $g^{-1} \in \mathcal{B}^*_+(V)$ by requiring
$$g^{-1} * g = \mathrm{id}\,.$$
(v) For $a \in \mathcal{B}(V)$ and $g \in \mathcal{B}_+(V)$ we define the operator $a^{\sharp_g} \in \mathcal{L}(V)$ by
$$a^{\sharp_g} = g^{-1} * a$$
(vi) For any bilinear form a on either V or V^* we denote by $\hat{a}$ the symmetrisation, i.e.
$$\hat{a}(v,w) = \frac{1}{2}\left(a(v,w) + a(w,v)\right).$$

3.2. *Remark.* For $a \in \mathcal{B}(V)$ and $g \in \mathcal{B}_+(V)$ we have
$$a(v,w) = g(a^{\sharp_g}(v), w) \quad \forall v, w \in V.$$

The following construction is very useful.

3.3. **Proposition.** *Let V be an n-dimensional real vector space, $\Omega \subset \mathcal{L}(V)$ open and F be as in Proposition 2.5. Define*
$$\begin{aligned}\Phi\colon \Lambda \subset \mathcal{B}_+(V) \times \mathcal{B}(V) &\to \mathbb{R}\\ (g,h) &\mapsto F(g^{-1} * \hat{h}),\end{aligned}$$
where Λ is the open subset such that $g^{-1} * \hat{h} \in \Omega$ for all $(g,h) \in \Lambda$. Then Φ is as smooth as F and the partial derivative of Φ at (g,h) with respect to h can be regarded as a symmetric bilinear form,
$$\frac{\partial \Phi}{\partial h}(g,h) \in \mathcal{B}^*(V).$$
Furthermore the derivatives of F and Φ are related by
$$\frac{\partial \Phi}{\partial h}(g,h)a = \mathrm{tr}(F'(g^{-1} * \hat{h}) \circ \hat{a}^{\sharp_g}) = dF(g^{-1} * \hat{h})\hat{a}^{\sharp_g}. \tag{3.1}$$

Proof. Since the map $h \mapsto g^{-1} * \hat{h}$ is linear, we obtain
$$\frac{\partial \Phi}{\partial h}(g,h)a = \mathrm{tr}\left(F' \circ (g^{-1} * \hat{a})\right)$$
and it can be regarded as a symmetric bilinear form acting on pairs (ξ, ζ) via letting it act on $\xi \otimes \zeta$. ☐

4. Properties of symmetric functions

We investigate some special properties associated to symmetric functions, which are particularly related to applications in geometric flows. The most crucial one, the monotonicity, usually ensures that a flow is parabolic. Define
$$\Gamma_+ = \{(\kappa_i) \in \mathbb{R}^n \colon \kappa_i > 0 \quad \forall 1 \leq i \leq n\}.$$

4.1. **Definition.** Let $\Gamma \subset \mathbb{R}^n$ open and symmetric, $r \geq 1$ and let $f \in C^r(\Gamma)$ be symmetric.

(i) f is called *strictly monotone*, if
$$\frac{\partial f}{\partial \kappa_i}(\kappa) > 0 \quad \forall \kappa \in \Gamma\ \forall 1 \leq i \leq n.$$
(ii) Let Γ in addition be a cone, then f is called *homogeneous of degree $p \in \mathbb{R}$* if
$$f(\lambda\kappa) = \lambda^p f(\kappa) \quad \forall \lambda > 0\ \forall \kappa \in \Gamma.$$

(iii) A nowhere vanishing function $f \in C^r(\Gamma_+)$, $r \geq 2$, is called *inverse concave (inverse convex)*, if the so-called *inverse symmetric function* $\tilde{f} \in C^r(\Gamma_+)$, defined by

$$\tilde{f}(\kappa_i) = \frac{1}{f(\kappa_i^{-1})},$$

is concave (convex).

These properties carry over to the function F from Proposition 2.5 in the following sense.

4.2. **Proposition.** *Let V be an n-dimensional real vector space, $\Gamma \subset \mathbb{R}^n$ open and symmetric, $r \geq 1$ and let $f \in C^r(\Gamma)$ and $F \in C^r(\Omega)$ be as in Proposition 2.5. Then there hold:*

(i) *If f is strictly monotone, then $F'(A)$ only has positive eigenvalues at all $A \in \mathcal{D}_\Gamma(V)$ and the bilinear form $\frac{\partial \Phi}{\partial h}$ from Proposition 3.3 is positive definite at all (g, h) with $g^{-1} * \hat{h} \in \mathcal{D}_\Gamma(V)$.*

(ii) *If Γ is a cone and f is homogeneous of degree p, then $\mathcal{D}_\Gamma(V)$ is a cone and $F_{|\mathcal{D}_\Gamma(V)}$ is homogeneous of degree p.*

(iii) *If $r \geq 2$, Γ is convex and f is concave, then F satisfies*

$$d^2F(A)(\eta, \eta) \leq 0$$

for all η having a symmetric matrix representation with respect to a basis of eigenvectors of A. The reverse inequality holds if f is convex.

Proof. (i) $F'(A)$ has positive eigenvalues due to Corollary 2.7. From (3.1) we obtain (omitting the arguments) for $0 \neq \xi \in V$,

$$\frac{\partial \Phi}{\partial h}(\xi, \xi) = \frac{\partial \Phi}{\partial h}(\xi \otimes \xi) = dF(\xi \otimes \xi)^{\sharp_g} > 0.$$

(ii) Let $A \in \mathcal{D}_\Gamma(V)$ and $\lambda > 0$. Then the claim follows from $\mathrm{EV}(\lambda A) = \lambda \mathrm{EV}(A)$.

(iii) Follows immediately from (2.8) and Lemma 2.10. $\square$

In Proposition 4.2, item (iii), the restriction to symmetric η is indeed necessary, as can be seen from Example 1.1

The following estimates for 1-homogeneous resp. inverse concave curvature functions are very useful and are also needed in [7]. The idea for the first statement comes from [1, Thm. 2.3] and also appeared in a similar form in [6, Lemma 14]. The proof for the second statement, however appearing in a slightly different form, can be found in [20, p. 112].

4.3. **Proposition.** *Let V be an n-dimensional real vector space and $r \geq 1$. Let $f \in C^r(\Gamma_+)$ and $F \in C^r(\Omega)$ be as in Proposition 2.5 with f symmetric, positive, strictly monotone and homogeneous of degree one. Then there hold:*

(i) *For every pair $A \in \mathcal{D}_{\Gamma_+}(V)$ and $g \in \mathcal{B}_+(V)$ such that A is self-adjoint with respect to g, there holds for all $\eta \in \mathcal{L}(V)$ that*

$$dF(A)(\mathrm{ad}_g(\eta) \circ A^{-1} \circ \eta) \geq F^{-1} \left(dF(A)\eta\right)^2,$$

where $\mathrm{ad}_g(\eta)$ is the adjoint of η with respect to g.

(ii) *If f is inverse concave, then for every pair $A \in \mathcal{D}_{\Gamma_+}(V)$ and $g \in \mathcal{B}_+(V)$ such that A is self-adjoint with respect to g, there holds*

$$d^2F(A)(\eta, \eta) + 2dF(A)(\eta \circ A^{-1} \circ \eta) \geq 2F^{-1} \left(dF(A)\eta\right)^2,$$

for all g-selfadjoint η.

Proof. (i) Note that for each $A \in \mathcal{D}_{\Gamma_+}(V)$ the kernel S of the map

$$dF(A)\colon \mathcal{L}(V) \to \mathbb{R}$$

has dimension $n^2 - 1$, due to the homogeneity which implies

$$dF(A)A = F(A) > 0.$$

Now let $\eta \in \mathcal{L}(V)$, then there exists a decomposition

$$\eta = aA + \xi,$$

where $\xi \in S$. Hence, omitting the argument A of F,

$$\begin{aligned} dF(\mathrm{ad}_g(\eta) \circ A^{-1} \circ \eta) &= adF(\eta) + adF(\mathrm{ad}_g(\xi)) + dF(\mathrm{ad}_g(\xi) \circ A^{-1} \circ \xi) \\ &\geq adF(\eta), \end{aligned}$$

since F' and A can be diagonalised simultaneously. The result follows from $F = dF(A) = a^{-1}dF(\eta)$.

(ii) For the inverse symmetric function $\tilde{f}$ the corresponding $\tilde{F}$ has the property

$$\tilde{F}(A) = \frac{1}{F(A^{-1})} \quad \forall A \in \mathcal{D}_{\Gamma_+}(V).$$

Thus we may differentiate $\tilde{F}$ using this formula, if we restrict to directions B which are self-adjoint with respect to g. Hence for all g-selfadjoint $A \in \mathcal{D}_{\Gamma_+}(V)$ we get

$$d\tilde{F}(A)B = \tilde{F}^2 dF(A^{-1})(A^{-1} \circ B \circ A^{-1})$$

and, omitting arguments,

$$\begin{aligned} d^2\tilde{F}(B,B) = 2\tilde{F}^3 &\left(dF(A^{-1} \circ B \circ A^{-1})\right)^2 \\ &- \tilde{F}^2 d^2F(A^{-1} \circ B \circ A^{-1}, A^{-1} \circ B \circ A^{-1}) \\ &- 2\tilde{F}^2 dF(A^{-1} \circ B \circ A^{-1} \circ B \circ A^{-1}), \end{aligned}$$

where $\tilde{F} = \tilde{F}(A)$ and $F = F(A^{-1})$. Since $\tilde{f}$ is inverse concave, there holds

$$d^2\tilde{F}(B,B) \leq 0$$

for all g-selfadjoint B. For some g-selfadjoint η set

$$B = A \circ \eta \circ A$$

to obtain

$$d^2F(\eta,\eta) + 2dF(\eta \circ A \circ \eta) \geq 2F^{-1}\left(dF(\eta)\right)^2,$$

where we again have in mind $F = F(A^{-1})$. The result follows. ☐

5. Examples

Let us have a look at some familiar symmetric functions, their corresponding associated operator functions and their properties. The most important examples are the elementary symmetric polynomials satisfying

$$s_k \circ \mathrm{EV}(A) = \frac{1}{k!}\frac{d^k}{dt^k}\det(I + tA)_{|t=0},$$

compare [11, equ. (2.1.31)]. s_k is strictly monotone on the set

$$\Gamma_k = \{\kappa \in \mathbb{R}^n \colon s_1(\kappa) > 0, \ldots, s_k(\kappa) > 0\},$$

which is equal to the connected component of the set $\{s_k > 0\}$ containing Γ_+, compare [13, Prop. 2.6]. Obviously s_1 is also concave and convex.

Define the quotients

$$q_k \colon \Gamma_{k-1} \to \mathbb{R}$$
$$q_k = \frac{s_k}{s_{k-1}}.$$

These are homogeneous of degree one and concave, cf. [13, Thm. 2.5]. On Γ_+ the q_k are also strictly monotone and inverse concave, cf. [1, Thm. 2.6]. Also the functions

$$f = \left(\frac{s_k}{s_l}\right)^{\frac{1}{k-l}}, \quad 0 \le l < k \le n,$$

share all these properties on Γ_+, [1, p. 23]. More examples of such curvature functions can be found in [1].

6. Loss of regularity

In this final section we discuss the regularity properties of the associated operator function F and show be means of an example that the loss of regularity from f to ψ in the correspondence

$$f = \psi(p_1, \dots, p_m)$$

also leads, in general, to the same loss of regularity from f to

$$F \colon \Omega \to \mathbb{R}$$

in the relation

$$F_{|\mathcal{D}_\Gamma(V)} = f \circ \mathrm{EV}_{|\mathcal{D}_\Gamma(V)}. \tag{6.1}$$

Consider the following example:

$$f(\kappa_1, \kappa_2) = (\kappa_1^2 + \kappa_2^2)^{\frac{3}{2}}.$$

Then $f \in C^2(\mathbb{R}^2)$. Since F is required to satisfy (6.1) and the *open* domain Ω of F has to contain the zero matrix, we must use

$$\psi(x_1, \dots, x_m) = |x_2|^{\frac{3}{2}}$$

to connect to f (note that $P_2(A)$ can be negative). Hence

$$F \colon \mathcal{L}(V) \to \mathbb{R}$$
$$F(A) = \psi(P_2(A)) = |\operatorname{tr}(A^2)|^{\frac{3}{2}}$$

is an associated operator function. Writing, with respect to a basis,

$$A = \begin{pmatrix} w & x \\ y & z \end{pmatrix},$$

we see that

$$F(A) = F(w, x, y, z) = |w^2 + 2xy + z^2|^{\frac{3}{2}},$$

which is not C^2, since its restriction to the straight line

$$x \mapsto (0, x, 1, 0) \tag{6.2}$$

is not C^2. It is in fact only as smooth as ψ. This is in sharp contrast to the regularity of the restriction to a subspace of g-selfadjoint operators,

$$F \colon \Sigma_g(V) \to \mathbb{R},$$

which has the same regularity as f, cf. [2, 19]. The crucial difference is that the variations in (6.2) are not allowed, since one must remain within the class of symmetric matrices.

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Appendix A2

PINCHING AND ASYMPTOTICAL ROUNDNESS FOR INVERSE CURVATURE FLOWS IN EUCLIDEAN SPACE

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PINCHING AND ASYMPTOTICAL ROUNDNESS FOR INVERSE CURVATURE FLOWS IN EUCLIDEAN SPACE

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ABSTRACT. We consider inverse curvature flows in the $(n+1)$-dimensional Euclidean space, $n \geq 2$, expanding by arbitrary negative powers of a 1-homogeneous, monotone curvature function F with some concavity properties. We obtain asymptotical roundness, meaning that circumradius minus inradius of the flow hypersurfaces decays to zero and that the flow becomes close to a flow of spheres.

1. INTRODUCTION

We consider inverse curvature flows in Euclidean space $\mathbb{R}^{n+1}$, $n \geq 2$,

$$\dot{x} = F^{-p}\nu, \; 0 < p < \infty, \tag{1.1}$$

where F is a symmetric, monotone, homogeneous of degree 1 and concave curvature function, which is defined in an open, convex cone of $\mathbb{R}^n$, such that F vanishes on its boundary, where in case $p > 1$ we assume $\Gamma = \Gamma_+$. Here ν is the outward normal to the flow hypersurfaces of the flow

$$x\colon [0, T^*) \times M \to \mathbb{R}^{n+1} \tag{1.2}$$

with starshaped initial hypersurface M_0, where in case $p > 1$ we assume that M_0 is strictly convex.

In [9] the same flow was considered. Here it is shown for any $0 < p < \infty$ that the maximal time of existence T^* is characterized by the property

$$\inf |x| \to \infty, \; t \to T^*, \tag{1.3}$$

and that the rescaled surfaces

$$\tilde{M}_t = \Theta^{-1} M_t, \tag{1.4}$$

where $\Theta = \Theta(t)$ is the radius of a suitable expanding sphere, converge to the unit sphere in C^∞, cf. [9, Thm. 1.1, Thm. 1.2].

The goal of this paper is the improvement of the asymptotical behavior of the flow. Let us explain our motivation to do this. To our knowledge, almost all of the existing results on classical smooth inverse curvature flows, cf. the end of this introduction for an overview, assume that the initial hypersurface may be written as a graph over a sphere. Of course, there are many possibilities to do this. For example take a sphere $\mathcal{S}$ centered at $q \in \mathbb{R}^{n+1}$ as initial hypersurface of the flow. It follows, that is evolves through expanding spheres

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centered at q as well. Now take another point $z \neq q$ in the interior of the ball enclosed by $\mathcal{S}$ and write $\mathcal{S}$ as a graph over a sphere $\mathcal{S}_z$ around z,

$$\mathcal{S} = \{(u(x), x) \colon x \in \mathcal{S}_z\}. \tag{1.5}$$

Let $\mathcal{S}$ evolve and let $u(t, \cdot)$ denote the corresponding graph functions over $\mathcal{S}_z$. From the previous observations it is clear that the oscillation of u can not decay to zero as $t \to T^*$, even though circumradius minus inradius of the hypersurfaces *is* constantly zero. Thus the choice of the sphere $\mathcal{S}_z$ is not optimal and u does not reflect the nice spherical shape of the evolving surfaces. The optimal sphere would be one around q. In this paper we are going to show that such an optimal sphere exists in the sense that the flow hypersurfaces will fit to a flow of spheres arbitrarily close. We are going to achieve this for $0 < p < \infty$ and for $n \geq 2$. The main ingredient in the proof is an estimate of the oscillation of the support function as it appears in [1, Prop. 4, Lemma 5], also cf. [17, Prop. 7.3]. These results hold for the case $n = 2$. In higher dimensions there is a generalization of these results, which provides closeness to a sphere in terms of the difference of the principal radii of a convex hypersurface, cf. [12, Thm. 1.4].

In a recent paper Hung and Wang came up with a counterexample to such an asymptotical roundness for hypersurfaces expanding by the inverse mean curvature flow in the hyperbolic space, cf. [10, Thm. 1]. This shows the impossibility of proving results like ours in $\mathbb{H}^{n+1}$.

Before we give an overview over previous results on expanding flows in Euclidean space, let us state the main result of this paper. We require the following assumptions on F.

1.1. **Assumption.** Let $\Gamma \subset \mathbb{R}^n$ be an open, convex and symmetric cone containing the positive cone

$$\Gamma_+ = \{(\kappa_i) \in \mathbb{R}^n \colon \kappa_i > 0,\ 1 \leq i \leq n\}. \tag{1.6}$$

Let F be a positive, monotone, symmetric and concave curvature function, normalized to $F(1, \dots, 1) = n$, such that

(i) in case $0 < p \leq 1$ we have $F \in C^\infty(\Gamma)$ and $F_{|\partial\Gamma} = 0$,

(ii) in case $p > 1$ we additionally have $\Gamma = \Gamma_+$.

Recall, that a hypersurface M is called *F-admissable*, if $F((\kappa_i)(x))$ is well-defined for all $x \in M$, where $\kappa_i(x)$ are the principal curvatures of M at x with respect to the inward unit normal.

The main result of this paper is the following one.

1.2. **Theorem.** *Let $n \geq 2$, $0 < p < \infty$ and let F satisfy Assumption 1.1. Let*

$$x_0 \colon M \hookrightarrow M_0 \subset \mathbb{R}^{n+1} \tag{1.7}$$

be the smooth embedding of a closed, orientable, connected and F-admissable hypersurface, which can be written as a graph over a sphere $\mathbb{S}^n$,

$$M_0 = \{(u(0, x), x) \colon x \in \mathbb{S}^n\}. \tag{1.8}$$

Then

(i) there exists a unique smooth solution on a maximal time interval

$$x \colon [0, T^*) \times M \hookrightarrow \mathbb{R}^{n+1}, \tag{1.9}$$

which satisfies the flow equation

$$\dot{x} = \frac{1}{F^p}\nu \\ x(0,\xi) = x_0(\xi), \tag{1.10}$$

where $\nu = \nu(t,\xi)$ *is the outward unit normal to* $M_t = x(t, M)$ *at* $x(t,\xi)$ *and* F *is evaluated at the principal curvatures of* M_t *at* $x(t,\xi)$.

(ii) *There exists a point* $Q \in \mathbb{R}^{n+1}$ *and a sphere* $S^* = S_{R^*}(Q)$ *around* Q *with radius* R^*, *such that the spherical solutions* S_t *with radii* R_t *of (1.10) with* $M_0 = S_{R^*}$ *satisfy*

$$\operatorname{dist}(M_t, S_t) \leq c R_t^{-\frac{p}{2}} \quad \forall t \in [0, T^*), \tag{1.11}$$

$c = c(p, M_0, F)$. *Here* dist *denotes the Hausdorff distance of compact sets.*

Statement (i) is just the existence of a solution on a maximal time interval. This result is not new, holds in even more general situations and a proof can be found in [7, Thm. 2.5.19, Lemma 2.6.1]. We stated it for convenience. Also note that the statement in (ii) indeed says that the flow becomes close to a flow of spheres. This is due to the fact that the radii of spheres which satisfy (1.10) with a sphere as initial hypersurface do converge to infinity during the maximal time of existence, cf. [9, Rem. 3.1] and [6, Thm. 0.1].

Indeed, (1.11) also allows to improve the rate of convergence of $\tilde{M}_t$ by choosing the *optimal* geodesic sphere to rescale. We will not carry this out here, but refer to [17, Sec. 7] for a rough outline of the arguments involved.

Now we give a brief overview over the state of the art in classical expanding curvature flows. We leave aside the theory of contracting flows, weak solutions, flows with boundary conditions, other ambient spaces and evolving curves, due to the tremendous amount of literature, which is not of direct interest with respect to our results.

For smooth, expanding flows in Euclidean space usually, except for [16], the asymptotic behavior of the flow hypersurfaces is described via the rescaling

$$\tilde{M}_t = \Theta^{-1} M_t, \tag{1.12}$$

where $\Theta(t)$ is the radius of the evolution of an *arbitrary* geodesic sphere, such that $\tilde{M}_t$ is bounded below and above. In those works, the authors show that $\tilde{M}_t$ converges to a sphere smoothly, which is less than (1.11) on the C^0-level.

Convergence of the rescaled surfaces $\tilde{M}_t$ was proven, for example, in the papers by Gerhardt, [6], and Urbas, [19], in the case $p = 1$ under similar assumptions on F as we do impose them. Results like these in the case $0 < p \leq 1$ and general F were derived in [20] and for more general, but still concave functions of the inverse Gauss curvature or the principal radii, compare the works by Chow and Tsai, [5] and [4] respectively, as well as [11]. Results for $p > 1$ have been accomplished for $n = p = 2$ and $F = 2K^{\frac{1}{2}}$, the classical inverse Gauss curvature flow in $\mathbb{R}^3$, by Schnürer, [16], and for $n = 2$, $1 < p < 2$ by Li, [13]. Probably the most general existing paper on classical inverse curvature flows in $\mathbb{R}^{n+1}$ is [9]. Besides those convergence results, Smoczyk has found an explicit representation for the solution of the inverse harmonic mean curvature flow, [18].

To our knowledge, the only situation, in which statement (ii) is proven, is the case $n = p = 2$ and $F = 2K^{\frac{1}{2}}$, where K is the Gaussian curvature, cf. [16]. We are not aware of the existence of a convergence result of type (1.11) in case of the other parameters.

2. Notation and definitions

In this article we consider closed, embedded and oriented hypersurfaces $M \hookrightarrow \mathbb{R}^{n+1}$, which can be written as graphs over a sphere $\mathbb{S}^n$,

$$M = \{(u(x), x) \colon x \in \mathbb{S}^n\}. \tag{2.1}$$

The coordinate representation $(u(x), x)$ is to be understood in polar coordinates, in which the Euclidean metric reads

$$d\bar{s}^2 = dr^2 + r^2\sigma_{ij}dx^i dx^j \equiv dr^2 + \bar{g}_{ij}dx^i dx^j. \tag{2.2}$$

We prefer the coordinate based notation for tensors.

Note that sometimes we use the slightly ambiguous notation to write x for an element $x \in \mathbb{S}^n$, where it is to be understood as $x = (x^i)$, latin indices ranging between 1 and n, or to write x as an element $x \in \mathbb{R}^{n+1}$, where it is then to be understood as $(x^\alpha) = (x^0, (x^i))$, greek indices ranging from 0 to n. Then x^0 denotes the radial component r.

We denote the standard induced metric of $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ by (σ_{ij}). The geometric quantities of M are denoted via the following notation. The induced metric is denoted by $g = (g_{ij})$ with inverse $g^{-1} = (g^{ij})$ and the second fundamental form with respect to the inward normal is $A = (h_{ij})$. Tensor indices of tensor fields on M are always lowered or lifted via g, unless stated otherwise, e.g.

$$h^i_j = g^{ik}h_{kj}. \tag{2.3}$$

Covariant derivatives with respect to the induced metric will simply be denoted by indices, e.g. u_i for a function $u \colon M \to \mathbb{R}$, or by a semicolon, if ambiguities are possible, e.g. $h_{ij;k}$.

The outward normal vector field to M is given by

$$(\nu^\alpha) = v^{-1}(1, -\check{u}^i), \tag{2.4}$$

where $\check{u}^i = \bar{g}^{ik}u_k$, $(\bar{g}^{ik}) = (\bar{g}_{ik})^{-1}$ and

$$v^2 = 1 + \bar{g}^{ij}u_i u_j \equiv 1 + |Du|^2. \tag{2.5}$$

For a tensor field $T = (t^{i_1,\dots i_k}_{j_1,\dots,j_l})$ on M the pointwise norm $\|T\|$ is always defined with respect to the induced metric

$$\|T\|^2 = t^{i_1,\dots,i_k}_{j_1,\dots,j_l} t^{j_1,\dots,j_l}_{i_1,\dots,i_k}. \tag{2.6}$$

A dot over a function or a tensor always denotes a total time derivative, e.g.

$$\dot{u} = \frac{d}{dt}u, \tag{2.7}$$

whereas a prime denotes differentiation with respect to a direct argument. If for example $f = f(u)$, then

$$f' = \frac{d}{du}f \tag{2.8}$$

and

$$\dot{f} = f'\dot{u}. \tag{2.9}$$

Note, that this notation partially deviates from those in [7] and [9].

Curvature functions.

The formulation of our assumptions on the curvature function F has used cones in $\mathbb{R}^n$, i.e. F depends smoothly on the principal curvatures,

$$F = F(\kappa_i). \tag{2.10}$$

However, as it is shown in [7, Ch. 2.1] and the references therein, it is also possible to consider F as a smooth function of the second fundamental form and the metric,

$$F = F(h_{ij}, g_{ij}), \tag{2.11}$$

or, as well, as a function defined on the mixed tensor (h^i_j),

$$F = F(h^i_j). \tag{2.12}$$

Those formulations are basically equivalent. In the formulation of evolution equations we will use the second of those three. Note, that

$$F^{kl} = \frac{\partial F}{\partial h_{kl}} \tag{2.13}$$

defines a tensor field on M of two contravariant indices.

Evolution equations.

2.1. **Remark.** The existence of a solution to (1.10) on a maximal time interval $[0, T^*)$ is well-known. We refer to [7, Thm. 2.5.19, Lemma 2.6.1]. Furthermore, the solution x exists at least as long as the solution

$$u \colon [0, \bar{T}) \times \mathbb{S}^n \to \mathbb{R} \tag{2.14}$$

of the scalar flow equation

$$\begin{aligned} \frac{\partial}{\partial t} u &= \frac{v}{F^p} \\ u(0, \cdot) &= u_0, \end{aligned} \tag{2.15}$$

where u_0 is the graph representation of the initial hypersurface, also compare [7, Thm. 2.5.17] and [7, p. 98-99]. Note as well that under Assumption 1.1 we have $\bar{T} = T^*$, cf. [9, Thm. 1.1, Thm. 1.2].

For real numbers $r > 0$ define

$$\Phi(r) = -r^{-p}. \tag{2.16}$$

The relevant evolution equations involved in the curvature flow are the following. The second fundamental form in mixed form satsifies

$$\dot{h}^i_j - \Phi' h^i_{j;kl} = \Phi' F^{kl} h_{rk} h^r_l h^i_j - (\Phi' F - \Phi) h^i_k h^k_j + \Phi^{kl,rs} h_{kl;j} h_{rs;}{}^i, \tag{2.17}$$

cf. [7, Lemma 2.4.1]. The curvature function Φ satisfies

$$\dot{\Phi} - \Phi' F^{kl} \Phi_{kl} = \Phi' F^{kl} h_{rk} h^r_l \Phi, \tag{2.18}$$

cf. [7, Lemma 2.3.4]. In the sequel we will need two other derived evolution equations, namely for the mean curvature $H = h^i_i$,

$$\dot{H} - \Phi' F^{kl} H_{kl} = \Phi' F^{kl} h_{rk} h^r_l H - (\Phi' F - \Phi) \|A\|^2 + \Phi^{kl,rs} h_{kl;i} h_{rs;}{}^i \tag{2.19}$$

and for $\|A\|^2 = h^i_j h^j_i$,

$$\begin{aligned}\frac{d}{dt}\|A\|^2 - \Phi' F^{kl}(\|A\|)_{kl} = 2\Phi' F^{kl} h_{rk} h^r_l \|A\|^2 - 2(\Phi' F - \Phi) h^i_k h^k_j h^j_i \\ + 2\Phi^{kl,rs} h_{kl;i} h_{rs;}{}^j h^i_j - 2\Phi' F^{kl} h^i_{j;k} h^j_{i;l}.\end{aligned} \tag{2.20}$$

2.2. **Remark.** For better readability of the subsequent results, we stick to the convention that whenever we claim the existence of constants, c, γ etc., they are allowed and understood to depend on n, p, M_0 and F as given data of the initial value problem, without mentioning this over and over again.

3. Pinching estimates

In this section we successively improve the pinching estimates. First we need to revisit some results from [9].

Let $\Theta = \Theta(t, r)$ denote a geodesic sphere with initial radius r, that exists as long as the solution x of (1.10) and for which

$$\tilde{u} = u\Theta^{-1} \tag{3.1}$$

is bounded below and above by positive constants, compare [9, Lemma 3.3-3.5] and [6, (4.8)]. The following proposition holds.

3.1. **Proposition.** *Let x be the solution of (1.10) under Assumption 1.1. Then the rescaled principal curvatures*

$$\tilde{\kappa}_i = \kappa_i \Theta \tag{3.2}$$

satisfy

$$0 < c^{-1} \le \tilde{\kappa}_i \le c \quad \forall t \in [0, T^*), \tag{3.3}$$

if $p > 1$, and in case $p \le 1$ the $\tilde{\kappa}_i$ stay within a compact subset of Γ.

Proof. For $p < 1$ this is [9, Lemma 4.11, Cor. 4.12] and for $p = 1$ we refer to [6, (4.9)]. In case $p > 1$ we refer to [9, Lemmata 3.10, 4.7, 4.9]. □

Furthermore, we obtain decay estimates for the gradient.

3.2. **Proposition.** *Let x be the solution of (1.10) under Assumption 1.1, $0 < p < \infty$. Then there exist positive constants c and γ, such that the function*

$$\varphi = \log u \tag{3.4}$$

satisfies the gradient estimate

$$|D\varphi| = \sigma^{ij}\varphi_i\varphi_j \le c\Theta^{-\gamma} \quad \forall t \in [0, T^*). \tag{3.5}$$

Proof. In case $p > 1$ this holds with $\gamma = \frac{1}{2}$, cf. [9, Lemma 3.7]. In case $p = 1$ this follows from [6, Lemma 2.5]. In case $p < 1$ this follows from the formula [9, (3.45)], where one should also note [9, (3.41)], as well as the spherical growth

$$\Theta(t, r) = \left(\frac{1-p}{n^p} t + r^{1-p}\right)^{\frac{1}{1-p}}, \tag{3.6}$$

cf. [9, (3.11)]. □

From these observations we deduce that the pinching of the hypersurfaces actually improves.

3.3. **Proposition.** *Let x be the solution of (1.10) under Assumption 1.1, $0 < p < \infty$. Then there exist positive constants c and γ, such that the principal curvatures κ_i of the flow hypersurfaces satisfy the pointwise estimate*

$$(\kappa_i - \kappa_j)^2 \le cH^{2+\gamma} \quad \forall t \in [0, T^*). \tag{3.7}$$

Proof. For this proof norms of tensors are formed with respect to σ_{ij}.

The function $\log \tilde{u}$ is bounded in $C^\infty(\mathbb{S}^n)$, cf. [9, Lemma 5.1, Thm. 5.2]. Thus, via interpolation, compare [8, Lemma 6.1] and Proposition 3.2, we obtain

$$|D^2\varphi|^2 \le c|D\varphi||\varphi|_{C^3} \le c\Theta^{-\gamma} \quad \forall t \in [0, T^*). \tag{3.8}$$

The rescaled second fundamental form $\tilde{h}^i_j = \Theta h^i_j$ satisfies

$$\tilde{h}^i_j = v^{-1}\tilde{u}^{-1}(\delta^i_j - (\sigma^{ik} - v^{-2}\varphi^i\varphi^k)\varphi_{kj}), \tag{3.9}$$

cf. [9, (5.2)] and [7, Lemma 2.7.6]. Thus

$$|\tilde{h}^i_j - \lambda\delta^i_j| \le c\Theta^{-\frac{\gamma}{2}}, \tag{3.10}$$

where we used Proposition 3.2 and the fact that $\tilde{u}$ converges to some constant λ^{-1}. Rescaling backwards yields

$$|\kappa_i - \lambda\Theta^{-1}| \le c\Theta^{-\left(1+\frac{\gamma}{2}\right)} \quad \forall 1 \le i \le n, \tag{3.11}$$

hence we obtain the result in view of

$$0 < c^{-1} \le \Theta H \le c. \tag{3.12}$$

□

The final pinching improvement will allow us to derive asymptotical roundness of the flow hypersurfaces.

3.4. **Proposition.** *Let x be the solution of (1.10) under Assumption 1.1, $0 < p < \infty$. Then for all $\delta < 4 + 2p$ there exists a constant $c = c_\delta > 0$, such that the principal curvatures κ_i satisfy the pointwise estimate*

$$(\kappa_i - \kappa_j)^2 \le cH^\delta \quad \forall t \in [0, T^*). \tag{3.13}$$

Proof. Using Proposition 3.3 it suffices to show this for $\delta > 2$. From (2.19) and (2.20) we obtain that

$$w = \|A\|^2 - \frac{1}{n}H^2 \tag{3.14}$$

satisfies

$$\begin{aligned}\dot{w} - \Phi' F^{kl} w_{kl} &= 2\Phi' F^{kl} h_{rk} h^r_l w - 2(\Phi' F - \Phi)\left(h^i_k h^k_j h^j_i - \frac{1}{n}\|A\|^2 H\right)\\ &\quad + 2\Phi^{kl,rs} h_{kl;i} h_{rs;}{}^j \left(h^i_j - \frac{1}{n}H\delta^i_j\right)\\ &\quad - 2\Phi' F^{kl}\left(h^i_{j;k} h^j_{i;l} - \frac{1}{n}H_k H_l\right)\end{aligned} \tag{3.15}$$

and thus, for positive constants c and $\delta > 2$ yet to be chosen, we obtain the evolution equation for

$$z = w - cH^{\delta}, \tag{3.16}$$

namely

$$\begin{aligned} \dot{z} - \Phi' F^{kl} z_{kl} = {} & 2\Phi' F^{kl} h_{rk} h_l^r z - c(\delta - 2)\Phi' F^{kl} h_{rk} h_l^r H^{\delta} \\ & - 2(\Phi' F - \Phi)\left(h_k^i h_j^k h_i^j - \frac{1}{n}\|A\|^2 H - \frac{c\delta}{2} H^{\delta-1}\|A\|^2\right) \\ & + 2\Phi^{kl,rs} h_{kl;i} h_{rs;}{}^j \left(h_j^i - \frac{1}{n} H \delta_j^i - \frac{c\delta}{2} H^{\delta-1}\delta_j^i\right) \\ & - 2\Phi' F^{kl}\left(h_{j;k}^i h_{i;l}^j - \frac{1}{n} H_k H_l\right) \\ & + c\delta(\delta-1) H^{\delta-2}\Phi' F^{kl} H_k H_l. \end{aligned} \tag{3.17}$$

From Proposition 3.3 we find $0 < t_0 < T^*$, such that

$$\max\left(\frac{w}{H^2}, \kappa_n\right) < \sigma := \frac{1}{\delta^{\alpha} n(n-1)} \quad \forall t \in [t_0, T^*), \tag{3.18}$$

where $\alpha_\delta >> 1$ will be chosen appropriately later. Thus t_0 will only depend on δ, p and M_0 as well and thus we may define

$$c = \frac{\sigma}{\inf\limits_{M_{t_0}} H^{\delta-2}}. \tag{3.19}$$

Note that due to [2, Lemma 2.2] the M_t are strictly convex for $t \geq t_0$. On M_{t_0} we have

$$z = w - cH^{\delta} = H^2\left(\frac{w}{H^2} - cH^{\delta-2}\right) < 0. \tag{3.20}$$

We wish to show that this remains valid up to T^*. Thus suppose $t_1 > t_0$ to be the first time, such that there exists $\xi_1 \in M_{t_1}$ with the property

$$z(t_1, \xi_1) = 0. \tag{3.21}$$

Define

$$\epsilon := cH^{\delta-2}(t_1, \xi_1), \tag{3.22}$$

then there holds

$$0 < \epsilon < \sigma, \tag{3.23}$$

due to (3.18). From (3.17) we obtain at (t_1, ξ_1) that

$$\begin{aligned} 0 \leq {} & -c(\delta-2)\Phi' F^{kl} h_{rk} h_l^r H^{\delta} \\ & - 2(\Phi' F - \Phi)\left(h_k^i h_j^k h_i^j - \frac{1}{n}\|A\|^2 H - \frac{c\delta}{2} H^{\delta-1}\|A\|^2\right) \\ & + 2\Phi^{kl,rs} h_{kl;i} h_{rs;}{}^j \left(h_j^i - \frac{1}{n} H \delta_j^i - \frac{c\delta}{2} H^{\delta-1}\delta_j^i\right) \\ & - 2\Phi' F^{kl}\left(h_{j;k}^i h_{i;l}^j - \frac{1}{n} H_k H_l\right) + c\delta(\delta-1) H^{\delta-2}\Phi' F^{kl} H_k H_l. \end{aligned} \tag{3.24}$$

From [2, Lemma 2.3] we have at (t_1, ξ_1), note that $w = \epsilon H^2$,

$$\begin{aligned} h^i_k h^k_j h^j_i - \left(\frac{1}{n} + \epsilon\right) H\|A\|^2 &\geq \epsilon \left(\frac{1}{n} + \epsilon\right)(1 - \sqrt{n(n-1)\epsilon})H^3 \\ &= \epsilon \left(1 - \sqrt{n(n-1)\epsilon}\right)\|A\|^2 H \\ &> 0 \end{aligned} \tag{3.25}$$

and we have

$$\begin{aligned} g^{kl}\left(h^i_{j;k} h^j_{i;l} - \frac{1}{n} H_k H_l\right) &= \left\| D\left(A - \frac{1}{n} H g\right)\right\|^2 \\ &\geq \frac{2(n-1)}{3n}\|DA\|^2 \\ &\geq \frac{2(n-1)}{n(n+2)}\|DH\|^2, \end{aligned} \tag{3.26}$$

cf. [2, Lemma 2.1]. In view of the concavity of F we have $F \leq H$, [7, Lemma 2.2.20], and thus from (3.24) we obtain

$$\begin{aligned} 0 \leq &-2(\Phi' F - \Phi)\left(\epsilon\left(2 - \sqrt{n(n-1)\epsilon}\right)\|A\|^2 H - \frac{\epsilon\delta}{2}\|A\|^2 H\right) \\ &- \epsilon(\delta - 2)\frac{p}{p+1}(\Phi' F - \Phi)\|A\|^2 H \\ &- \epsilon(\delta - 2)\frac{p}{p+1}(\Phi' F - \Phi)(F^{kl} - g^{kl}) h_{rk} h^r_l H \\ &+ 2H\Phi^{kl,rs} h_{kl;i} h_{rs;}{}^j \left(H^{-1} h^i_j - \frac{1}{n}\delta^i_j - \frac{\epsilon\delta}{2}\delta^i_j\right) - \frac{2(n-1)}{n(n+2)}\Phi'\|DH\|^2 \\ &- \frac{2(n-1)}{3n}\Phi'\|DA\|^2 - 2\Phi'(F^{kl} - g^{kl})\left(h^i_{j;k} h^j_{i;l} - \frac{1}{n} H_k H_l\right) \\ &+ \epsilon\delta(\delta - 1)\Phi'\|DH\|^2 + \epsilon\delta(\delta - 1)\Phi'(F^{kl} - g^{kl}) H_k H_l. \end{aligned} \tag{3.27}$$

In view of (3.18) and due to to the fact that

$$\|F^{kl} - g^{kl}\| \to 0, \tag{3.28}$$

we may without loss of generality enlarge t_0 and α, such that the terms involving curvature derivatives are absorbed by the terms

$$-\frac{2(n-1)}{n(n+2)}\Phi'\|DH\|^2 \text{ and } -\frac{2(n-1)}{3n}\Phi'\|DA\|^2. \tag{3.29}$$

In this case at (t_1, ξ_1) we obtain

$$\begin{aligned} 0 \leq &-2(\Phi' F - \Phi)\epsilon\|A\|^2 H\left(\frac{p+2}{p+1} - \delta^{-\frac{\alpha}{2}} - \frac{\delta}{2(p+1)}\right) \\ &- \epsilon(\delta - 2)\frac{p}{p+1}(\Phi' F - \Phi)(F^{kl} - g^{kl}) h_{rk} h^r_l H \\ &< 0, \end{aligned} \tag{3.30}$$

if we choose

$$\delta < 4 + 2p, \tag{3.31}$$

$\alpha = \alpha(p, \delta)$ large enough and again t_0 larger to ensure that the term involving $\|F^{kl} - g^{kl}\|$ can be absorbed by the strictly negative, remaining part of the first line in (3.30). This contradiction shows that z will remain negative up to the time T^*, which yields the result. □

4. Oscillation decay

Let $\rho_+(t)$ denote the *circumradius* of M_t, i.e. the radius of the smallest ball in $\mathbb{R}^{n+1}$ enclosing M_t and analogously $\rho_-(t)$ the *inradius* of M_t, the radius of the largest ball in $\mathbb{R}^{n+1}$ enclosed by M_t. In this section we prove (ii) of Theorem 1.2, namely that $\rho_+ - \rho_-$ converges to zero and that we find an expanding family of geodesic spheres S_t with radii R_t the flow hypersurfaces M_t fit themselves to,

$$\operatorname{dist}(M_t, S_t) < cR_t^{-\frac{p}{2}}. \tag{4.1}$$

Note that the R_t represent a suitable choice of the $\Theta(t)$.

Let $\hat{M}_t$ denote the convex body enclosed by M_t, which is well defined for large t. For an interior point $y \in \operatorname{int}(M_t)$ let u_y denote the graph representation of M_t over the standard sphere centered in y,

$$M_t = \{(u_y(x), x) \colon x \in \mathbb{S}^n(y)\}. \tag{4.2}$$

4.1. **Proposition.** *Let x be the solution of (1.10) under Assumption 1.1, $0 < p < \infty$. Then there holds*

$$\rho_+(t) - \rho_-(t) \le \operatorname{osc} u_{y_t} \le c\Theta^{-\frac{p}{2}}(t) \quad \forall t \in [0, T^*), \tag{4.3}$$

where y_t is a suitable oscillation minimizing center of $\hat{M}_t$.

Proof. First note that Θ is still an arbitrary rescaling factor as in Proposition 3.1.

From (3.13) we deduce that principal curvatures satisfy

$$(\kappa_i - \kappa_j)^2 \le cH^\delta \le c\Theta^{-\delta} \tag{4.4}$$

for $\delta = 4 + p$. Since for large t the M_t are strictly convex we may consider the difference of the largest and smallest principal radius of curvature,

$$\left|\frac{1}{\kappa_1} - \frac{1}{\kappa_n}\right| = \left|\frac{\kappa_n - \kappa_1}{\kappa_1\kappa_n}\right| \le c\Theta^{2-\frac{\delta}{2}}. \tag{4.5}$$

Applying [12, Thm. 1.4] we obtain

$$\operatorname{dist}(M_t, S(y_t)) \le c\Theta^{2-\frac{\delta}{2}}, \tag{4.6}$$

from which the claim follows. Note that if a y_t happens not to minimize the oscillation, we can adjust it to do so. A simple geometric argument also shows that then the y_t must lie in the convex body $\hat{M}_t$. □

In order to find an optimally fitted spherical flow, we need the centers y_t from Proposition 4.1 to converge.

4.2. **Lemma.** *The centers y_t from Proposition 4.1 converge in $\mathbb{R}^{n+1}$. In particular we have*

$$|y_t - Q| \le c\Theta^{-p}(t) \tag{4.7}$$

for the limit $Q \in \mathbb{R}^{n+1}$.

Proof. For a point $q \in \mathbb{R}^{n+1}$ let

$$\bar{u}_q = \langle x - q, \nu \rangle \tag{4.8}$$

denote the support function with respect to q. For t close to T^* the M_t are strictly convex and hence we may apply a gradient estimate for convex hypersurfaces, [7, Lemma 2.7.10], to conclude

$$v \le e^{\bar{\kappa} \operatorname{osc} u_{y_t}}, \tag{4.9}$$

where $\bar{\kappa}$ is an upper for the principal curvatures of the slices $\{x^0 = \text{const}\}$, which in our case can be estimated:

$$\bar{\kappa} \le \max_{M_t} \frac{1}{u_{y_t}}. \tag{4.10}$$

Thus

$$v - 1 \le e^{c\bar{\kappa}\Theta^{-\frac{p}{2}}} - 1 \le c\bar{\kappa}\Theta^{-\frac{p}{2}}. \tag{4.11}$$

Since

$$\frac{u_{y_t}}{v} = \bar{u}_{y_t}, \tag{4.12}$$

we obtain

$$|\bar{u}_{y_t} - u_{y_t}| \le u_{y_t} \frac{v-1}{v} \le c\Theta^{-\frac{p}{2}} \tag{4.13}$$

and hence

$$\max \bar{u}_{y_t} - \min \bar{u}_{y_t} \le \max u_{y_t} - \min u_{y_t} + c\Theta^{-\frac{p}{2}} \le c\Theta^{-\frac{p}{2}}(t). \tag{4.14}$$

Using that the oscillation of the support function with respect to a fixed point is decreasing, cf. [14, Thm. 3.1], originally proved by Chow and Gulliver, [3], we obtain

$$\operatorname{osc} u_{y_{t_2}} \le \operatorname{osc} \bar{u}_{y_{t_2}} \le \operatorname{osc} \bar{u}_{y_{t_1}} \le c\Theta^{-\frac{p}{2}}(t_1) \quad \forall t_2 > t_1 \tag{4.15}$$

and since y_{t_2} is oscillation minimizing we must have

$$|y_{t_1} - y_{t_2}| \le c\Theta^{-\frac{p}{2}}(t_1) \quad \forall t_2 > t_1. \tag{4.16}$$

Letting $t_1 \to T^*$ we obtain the limit Q and then for fixed t_1 letting $t_2 \to T^*$, we obtain the desired estimate. ☐

We finish this paper by showing that the flow actually becomes close to a flow of spheres.

4.3. **Theorem.** *Let x be the solution of (1.10) under Assumption 1.1. Then there exists a geodesic sphere $S_{R^*}(Q)$, where Q is the limit from Lemma 4.2, such that the spherical leaves S_t of the initial value problem*

$$\begin{aligned} \dot{y} &= \frac{1}{F^p}\nu \\ y(0, M) &= S_{R^*}(Q) \end{aligned} \tag{4.17}$$

and the flow hypersurfaces M_t of the flow x satisfy

$$\operatorname{dist}(M_t, S_t) < cR_t^{-\frac{p}{2}}, \tag{4.18}$$

where R_t is the radius of S_t.

Proof. In case $p > 1$, R^* is determined by the requirement, that the spherical flow exists as long as the flow x. Then for all large t we must have

$$S_t \cap M_t \neq \emptyset, \tag{4.19}$$

compare the arguments at the end of the proof of [16, Lemma 5.1]. The result follows from the propositions 4.1 and 4.2.

In case $p \leq 1$ we need a different argument to show that there exists an expanding flow of spheres with the property (4.19), since we have $T^* = \infty$ and R^* is not determined clearly. Recall (3.6), that for initial radius r the radius R of a sphere evolves according to

$$R(t, r) = \left(\frac{1-p}{n^p} t + r^{1-p}\right)^{\frac{1}{1-p}}, \tag{4.20}$$

if $p < 1$ and

$$R(t, r) = re^{\frac{t}{n}}, \tag{4.21}$$

if $p = 1$. We see that in any case

$$R_t = R(t, \cdot) \tag{4.22}$$

is an increasing diffeomorphism from $(0, \infty)$ onto its image $\left((\frac{1-p}{n^p}t)^{\frac{1}{1-p}}, \infty\right)$ in case $p < 1$ and onto $(0, \infty)$ in case $p = 1$. Now let $u = u_Q$ and define sequences

$$\bar{R}^k = R_k^{-1}(\sup u(k, \cdot)), \tag{4.23}$$

$$\bar{R}_k = R_k^{-1}(\inf u(k, \cdot)) \tag{4.24}$$

and

$$R^k = \frac{1}{2}(\bar{R}^k + \bar{R}_k). \tag{4.25}$$

By the maximum principle $\bar{R}^k$ is non-increasing. There holds

$$S_k^k \cap M_k \neq \emptyset, \tag{4.26}$$

where S_k^k denotes the spherical leave at time k, which has started with initial radius R^k. Since in case $p < 1$

$$\frac{d}{dr} R_k(r) = \left(\frac{1-p}{n^p} k + r^{1-p}\right)^{\frac{p}{1-p}} r^{-p} \tag{4.27}$$

and in case $p = 1$

$$\frac{d}{dr} R_k(r) = e^{\frac{k}{n}}, \tag{4.28}$$

$\frac{d}{dr} R_k$ is uniformly bounded from below and since

$$\operatorname{osc} u \to 0, \tag{4.29}$$

we obtain that $\bar{R}^k$, $\bar{R}_k$ and R^k all converge to the same limit R^*. We claim, that the initial sphere S_{R^*} around Q leads to a spherical flow satisfying (4.19).

Otherwise there existed a time k_0, such that without loss of generality

$$R(k_0, R^*) < \inf u(k_0, \cdot). \tag{4.30}$$

By continuity of ODE orbits with respect to initial values on compact intervals, there is $\tilde{R} > R^*$, such that

$$R(k, R^*) < R(k, \tilde{R}) < \inf u(k, \cdot) \quad \forall k \geq k_0, \tag{4.31}$$

where we also used the maximum principle. Applying R_k^{-1} we find

$$R^* < \tilde{R} < \bar{R}_k \quad \forall k \geq k_0, \tag{4.32}$$

since R_k^{-1} is increasing with respect to r. This is a contradiction to $\bar{R}_k \to R^*$. So the spherical leaves of the flow with initial value $S_{R^*}(Q)$ intersect the M_t for all large times and due to the oscillation estimates we obtain the desired result. □

5. Concluding remarks

The pinching estimates, Proposition 3.4, turned out to improve, whenever p becomes larger. This fact is somehow surprising, since the evolution equation of the gradient function

$$v^2 = 1 + |Du|^2 \tag{5.1}$$

does not allow to apply the classical maximum principle, also compare the proof of [9, Lemma 3.6], so one could expect that it should become harder to control oscillations. As we have seen, however, the equation for the traceless second fundamental form serves as a way out.

Also note that in a further work we applied a method similar to the one in section 4 to prove that there can not be an estimate of the form

$$\operatorname{dist}(M, S_R) \leq c\|\mathring{A}\|^{\alpha}, \quad \alpha > 1, \tag{5.2}$$

in the class of uniformly convex hypersurfaces with a universal constant. The idea is that otherwise we could use a similar proof as in section 4 to prove asymptotical roundness of the inverse mean curvature flow in the hyperbolic space, which has shown not to be true in [10]. See [15] for a preprint version and a detailed description of this result.

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Appendix A3

EXPLICIT RIGIDITY OF ALMOST-UMBILICAL HYPERSURFACES

by Julien Roth and Julian Scheuer
to appear in *Asian Journal of Mathematics*

EXPLICIT RIGIDITY OF ALMOST-UMBILICAL HYPERSURFACES

JULIEN ROTH AND JULIAN SCHEUER

Abstract. We give an explicit estimate of the distance of a closed, connected, oriented and immersed hypersurface of a space form to a geodesic sphere and show that the spherical closeness can be controlled by a power of an integral norm of the traceless second fundamental form, whenever the latter is sufficiently small. Furthermore we use the inverse mean curvature flow in the hyperbolic space to deduce the best possible order of decay in the class of C^∞-bounded hypersurfaces of the Euclidean space.

1. Introduction

In this paper we prove two stability theorems of almost-umbilicity type, which give an answer to a question raised in [13] and thereby partially improve [9, Thm. 1.3, Thm. 1.4]. Furthermore we use a recent counterexample for the inverse mean curvature flow in the hyperbolic space, cf. [10], to provide a new counterexample for spherical closeness estimates.

Let us shortly introduce the relevant notation. For an oriented hypersurface of a Riemannian manifold, $M^n \hookrightarrow N^{n+1}$, g denotes its induced metric, $|M|$ its surface area, A its second fundamental form, $\mathring{A}$ the traceless part of A,

$$\mathring{A} = A - Hg, \tag{1.1}$$

x_M the center of mass of M and $d_{\mathcal{H}}$ the Hausdorff distance of sets.

For a tensor field $(T^{j_1\dots j_l}_{i_1\dots i_k})$ on M, we define its L^p-norm to be

$$\|T\|_p = \left(\int_M |T^{j_1\dots j_l}_{i_1\dots i_k} T^{i_1\dots i_k}_{j_1\dots j_l}|^{\frac{p}{2}}\right)^{\frac{1}{p}}, \tag{1.2}$$

where indices are raised or lowered with the help of g. Let us formulate our first main result.

1.1. **Theorem.** *Let $M \hookrightarrow \mathbb{R}^{n+1}$ be a closed, connected, oriented and immersed C^2-hypersurface with $|M| = 1$. Let $p > n \geq 2$. Then there exist constants $c, \epsilon_0 > 0$ depending on n, p and $\|A\|_p$, as well as a constant $\alpha = \alpha(n,p)$, such that whenever there holds*

$$\|\mathring{A}\|_p < \|H\|_p \epsilon_0, \tag{1.3}$$

then

$$d_{\mathcal{H}}(M, S_R(x_M)) \leq \frac{c^\alpha R}{\|H\|_p^\alpha} \|\mathring{A}\|_p^\alpha \equiv R\epsilon^\alpha \tag{1.4}$$

and M is ϵ^α-quasi-isometric to a sphere S_R with a certain radius R.

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1.2. *Remark.* (i) By ϵ^α-quasi-isometric we mean that a suitable diffeomorphism F from M into S_R satisfies

$$|d(F(x_1), F(x_2)) - d(x_1, x_2)| \leqslant R\epsilon^\alpha \tag{1.5}$$

for any $x_1, x_2 \in M$.

(ii) The radius R can be expressed in terms of $\|H\|_p$, compare [15, Cor. 4.6].

(iii) The assumption $|M| = 1$ is only for simplification. By scaling it is easy to obtain a scale-invariant version for arbitrary volume.

In Section 3 we generalize this theorem to conformally flat ambient spaces.

The history of the problem to control the closeness to a sphere by curvature quantities is quite long, starting from the well known *Nabelpunktsatz*. We refer to the bibliography in [13] for a quite detailed overview. Let us only mention several results which have appeared recently. For surfaces, $n = 2$, a quite straightforward calculation due to Andrews yields an explicit C^0-estimate for convex hypersurfaces, cf. [1, Prop. 4, Lemma 5],

$$\left|\langle x - q, \nu\rangle - \frac{1}{8\pi}\int_M H\right| \leq C|M|\|\mathring{A}\|_\infty, \tag{1.6}$$

where x is the embedding vector and q is the Steiner point. In Section 4 we use the inverse mean curvature flow (IMCF) in the hyperbolic space to prove that the power on the right-hand side of (1.6) can not be improved to $\|\mathring{A}\|^\alpha_\infty$, $\alpha > 1$, which is in turn then not possible either in Theorem 1.1. The latter proof relies on a recent example due to Hung and Wang, [10, Thm. 1, Prop. 5], that the convergence after rescaling in the IMCF can not be too fast in the hyperbolic space.

For strictly convex hypersurfaces of $\mathbb{R}^{n+1}$ there is the following estimate of circumradius R minus inradius r due to Leichtweiß, cf. [11, Thm. 1.4, eq. (38)]:

$$R - r \leq c_n \max_{x\in M}(R_n(x) - R_1(x)), \tag{1.7}$$

where $R_1 \leq \cdots \leq R_n$ are the ordered radii of curvature. Theorem 1.1 deals with estimates in dependence of integral pinching. For the case $n = 2$, an estimate similar to (1.4) with a better constant was obtained by De Lellis and Müller, cf. [4]

In [13, Cor. 1.2] Perez derived a *qualitative* solution and obtained under certain assumptions, for given $\epsilon > 0$, a $\delta > 0$, such that

$$\|\mathring{A}\|_p < \delta \tag{1.8}$$

implies

$$d_{\mathcal{H}}(M, S_{r_0}(x)) < \epsilon. \tag{1.9}$$

In [13, p. xvi] the author posed the derivation of an explicit δ as a question of interest.

Note that in (1.4) we did not achieve a constant independent of the size of the curvature itself. The constant is only uniform in the class of hypersurfaces with a fixed bound on the curvature of the hypersurface.

The following theorem, due to Grosjean and the first author, [9, Thm. 1.4], already provides this conclusion, however only with the additional assumption of smallness of the oscillation of the mean curvature itself:

1.3. **Theorem.** [9, Thm. 1.4]
Let (M^n, g) be a compact, connected and oriented n-dimensional Riemannian manifold without boundary isometrically immersed by ϕ in $\mathbb{R}^{n+1}$. Let $\epsilon < 1$, $r, q > n$, $s \geq r$ and $c > 0$. Let us assume that $|M|^{\frac{1}{n}} \|H\|_q \leq c$. Then there exist positive constants $C = C(n, q, c)$, $\alpha = \alpha(q, n)$, such that if $\epsilon^\alpha \leq \frac{1}{C}$,

$$\|\mathring{A}\|_r \leq \|H\|_r \epsilon \tag{1.10}$$

and

$$\|H^2 - \|H\|_s^2\|_{\frac{r}{2}} \leq \|H\|_r^2 \epsilon, \tag{1.11}$$

then M is ϵ^α-Hausdorff close to $S_{\frac{1}{\|H\|_2}}(x_M)$. Moreover if $|M|^{\frac{1}{n}} \|A\|_q \leq c$, then M is diffeomorphic and ϵ^α-quasi-isometric to $S_{\frac{1}{\|H\|_2}}(x_M)$.

Note that in this theorem, L^p-norms are defined slightly different, namely such that the L^p-norms of scale-invariant functions are scale-invariant. Our notation corresponds to the one in [13]. This ambiguity does not cause any problems, since we prove Theorem 1.1 for $|M| = 1$. Also note the typo in [9, Thm. 1.4], where the α is missing in the conclusion.

In [14, Thm. 3.1], which also covers other ambient spaces, (1.11) was replaced by an assumption on the gradient of H. However, with the help of the following theorem due to Perez it is possible to get rid of (1.11) completely.

1.4. **Theorem.** [13, Thm. 1.1]
Let $p > n \geq 2$ and $c_0 > 0$ be given. Then there is a constant $C > 0$, depending only on n, p and c_0, such that:
If $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth, closed and connected n-dimensional hypersurface with

$$|\Sigma| = 1 \tag{1.12}$$

and

$$\|A\|_p \leq c_0, \tag{1.13}$$

then

$$\min_{\lambda \in \mathbb{R}} \|A - \lambda g\|_p \leq C \|\mathring{A}\|_p. \tag{1.14}$$

The proof of Theorem 1.1 is a combination of Theorem 1.3 and Theorem 1.4.

2. Proofs of Theorem 1.1

Proof no. 1: Without loss of generality we may suppose that M is of class C^∞, since both sides of the inequality are continuous with respect to the C^2-norm and hence the general result can then be achieved by approximation.

Using Theorem 1.4, we obtain a $\lambda_0 \in \mathbb{R}$, such that

$$\|A - \lambda_0 g\|_p \leq C' \|\mathring{A}\|_p, \tag{2.1}$$

where $C' = C'(n, p, \|A\|_p)$. Let us calculate

$$
\begin{aligned}
\|H^2 - \|H\|_p^2\|_{\frac{p}{2}} &\leq \|H^2 - \lambda_0^2\|_{\frac{p}{2}} + \|\lambda_0^2 - \|H\|_p^2\|_{\frac{p}{2}} \\
&= \left(\int_M |H-\lambda_0|^{\frac{p}{2}}|H+\lambda_0|^{\frac{p}{2}}\right)^{\frac{2}{p}} + |\lambda_0^2 - \|H\|_p^2| \\
&\leq 2(\|H\|_p + |\lambda_0|)\|H-\lambda_0\|_p \\
&\leq c_n(\|H\|_p + |\lambda_0|)\|A-\lambda_0 g\|_p \\
&\leq c'\|H\|_p\|\mathring{A}\|_p,
\end{aligned}
\tag{2.2}
$$

where $c' = c'(n, p, \|A\|_p)$. The last inequality is due to the fact that

$$
|\lambda_0 - \|H\|_p| \leq c''\|\mathring{A}\|_p. \tag{2.3}
$$

Defining

$$
c = \max(1, c'), \tag{2.4}
$$

$$
\epsilon = \frac{c\|\mathring{A}\|_p}{\|H\|_p}, \tag{2.5}
$$

and

$$
\epsilon_0 := \frac{\min\left(1, C^{-\frac{1}{\alpha}}\right)}{2c} \tag{2.6}
$$

then by (1.3),

$$
\epsilon \leq c\epsilon_0 = \frac{1}{2}\min\left(1, C^{-\frac{1}{\alpha}}\right), \tag{2.7}
$$

where α and C are the constants from Theorem 1.3. Furthermore we have

$$
\|\mathring{A}\|_p \leq \|H\|_p\epsilon \tag{2.8}
$$

and

$$
\|H^2 - \|H\|_p^2\|_{\frac{p}{2}} \leq \|H\|_p^2\epsilon. \tag{2.9}
$$

Thus we may apply Theorem 1.3 to conclude that M is ϵ^α-close to a sphere.

The proof of the theorem we applied here, Theorem 1.3, relies on a pinching result for the first eigenvalue which was proven in [9] for a much more general class of ambient spaces. Thus it might not be easily accessible from our point of view. For convenience we want to repeat their main steps of the proof of this theorem in our Euclidean setting, see [9, p. 487] for the original one. For this purpose we use a recent pinching result for the first eigenvalue of the Laplace operator by both of the authors, cf. [15, Thm. 1.1]. This, and also the original proof in [9], uses the fact that pinching of the Ricci tensor can be controlled by pinching of the traceless second fundamental form. Then we apply an eigenvalue pinching result due to Aubry, which was proved in [2, Prop. 1.5] and can also be found in [3, Thm. 1.6]. It says that for $p > n/2$, a complete Riemannian manifold (M^n, g) with

$$
\frac{1}{|M|}\int_M (\underline{\mathrm{Ric}} - (n-1))_-^p < \frac{1}{C(p,n)} \tag{2.10}
$$

is compact and satisfies

$$\lambda_1 \geqslant n\left(1 - C(n,p)\left(\frac{1}{|M|}\int_M (\underline{\mathrm{Ric}} - (n-1))_-^p\right)^{\frac{1}{p}}\right), \tag{2.11}$$

where $\underline{\mathrm{Ric}}$ denotes the smallest eigenvalue of the Ricci tensor and

$$x_- = \max(0, -x). \tag{2.12}$$

Proof no. 2: Due to the Gauss equation and a simple calculation we obtain a formula for the Ricci tensor in terms of the second fundamental form, namely we obtain

$$R_{ij} - (n-1)H^2 g_{ij} = (n-2)H(h_{ij} - Hg_{ij}) - (h_{ik} - Hg_{ik})(h_j^k - H\delta_j^k). \tag{2.13}$$

Thus

$$\begin{aligned} \|\mathrm{Ric} - (n-1)\|H\|_p^2 g\|_{\frac{p}{2}} &\le c\|H\|_p\|\mathring{A}\|_p + c\|\mathring{A}\|_{\frac{p}{2}}^2 + \|H^2 - \|H\|_p^2\|_{\frac{p}{2}} \\ &\le c\|H\|_p\|\mathring{A}\|_p, \end{aligned} \tag{2.14}$$

where we used (2.2) and $c = c(n,p,\|A\|_p)$. Using a scaled version of Aubry's eigenvalue estimate we obtain the existence of a constant $\epsilon_0 = \epsilon_0(n,p,\|A\|_p)$, such that

$$\|\mathring{A}\|_p \le \epsilon_0\|H\|_p \tag{2.15}$$

implies

$$\begin{aligned} \lambda_1 &\ge n\left(\|H\|_p^2 - c\|\mathrm{Ric} - (n-1)\|H\|_p^2 g\|_{\frac{p}{2}}\right) \\ &\ge n\|H\|_p^2 - c\|H\|_p^2\frac{\|\mathring{A}\|_p}{\|H\|_p} \\ &\ge \left(1 - c\frac{\|\mathring{A}\|_p}{\|H\|_p}\right)n\|H\|_p^2. \end{aligned} \tag{2.16}$$

Now we can apply the abstract eigenvalue pinching result [15, Thm. 2], applied to the tensors $S = T = \mathrm{id}$.

3. Generalization to conformally flat manifolds

Using that the property of a hypersurface to be totally umbilic is invariant with respect to a conformal change of the ambient metric, we easily obtain the following generalization to conformally flat manifolds, which in particular include the half-sphere and the hyperbolic space and improves the ϵ^α-proximity statement in [9, Thm. 1.3] in the sense that it removes an assumption similar to (1.11).

3.1. **Theorem.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be open and let $N^{n+1} = (\Omega, \bar{g})$ be a conformally flat Riemannian manifold, i.e.*

$$\bar{g} = e^{2\psi}\tilde{g}, \tag{3.1}$$

where $\tilde{g}$ is the Euclidean metric and $\psi \in C^\infty(\Omega)$. Let $M^n \hookrightarrow N^{n+1}$ be a closed, connected, oriented and immersed C^2-hypersurface. Let $p > n \ge 2$. Then there exist constants c and ϵ_0, depending on n, p, $|M|$, $\|\tilde{A}\|_p$ and $\|\psi\|_{\infty,M}$, as well as a constant $\alpha = \alpha(n,p)$, such that whenever there holds

$$\|\mathring{A}\|_p \le \|\tilde{H}\|_p\epsilon_0, \tag{3.2}$$

there also holds

$$d_{\mathcal{H}}(M, S_R) \leq \frac{cR}{\|\tilde{H}\|_p^\alpha} \|\mathring{A}\|_p^\alpha, \tag{3.3}$$

where S_R is the image of a Euclidean sphere considered as a hypersurface in N^{n+1}, $\|\tilde{A}\|_p$ and $\|\tilde{H}\|_p$ are the corresponding Euclidean quantities and the Hausdorff distance is measured with respect to the metric $\bar{g}$.

3.2. *Remark.* Since in conformally flat spaces the scaling behaviour of the second fundamental form with respect to homotheties heavily depends on the nature of the ambient space, in this case there seems to be no way to give a general scale invariant estimate. This is the reason why this closeness estimate is only uniformly valid in the class of C^2-bounded hypersurfaces.

Furthermore note that for example in all simply connected space forms the hypersurface S_R is actually a geodesic sphere. This follows from the fact that in those spaces totally umbilical hypersurfaces are spheres and total umbilicity invariant with respect to conformal transformations of the ambient space, as will be apparent from the following proof of Theorem 3.1.

Thus Theorem 3.1 gives an explicit spherical closeness estimate of almost-umbilical hypersurfaces in the hyperbolic space as well as in the half-sphere of constant positive sectional curvature.

Proof. Under a conformal relation of the metrics as in (3.1) the corresponding induced geometric quantities of the the embedded hypersurface M are related as follows.

$$g_{ij} = e^{2\psi} \tilde{g}_{ij} \tag{3.4}$$

and

$$h_{ij} e^{-\psi} = \tilde{h}_{ij} + \psi_\beta \tilde{\nu}^\beta \tilde{g}_{ij}, \tag{3.5}$$

where $\tilde{\nu}$ is the normal to M. Those formulae can be found in [7, Prop. 1.1.11]. Hence

$$h_{ij} - H g_{ij} = e^{\psi} (\tilde{h}_{ij} - \tilde{H} \tilde{g}_{ij}) \tag{3.6}$$

and hence

$$c\|\mathring{\tilde{A}}\|_p \leq \|\mathring{A}\|_p \leq C\|\mathring{\tilde{A}}\|_p, \tag{3.7}$$

where the constants depend on $\|\psi\|_{\infty,M}$. Since the Euclidean and the conformal Hausdorff distances are equivalent whenever $|\psi|$ is bounded, we obtain the result after applying Theorem 1.1. ☐

Due to a well known interpolation theorem for convex hypersurfaces of Riemannian manifolds we obtain the following gradient stability estimate in space forms.

3.3. **Corollary.** *Let N^{n+1} be the Euclidean space, the hyperbolic space or the sphere. Let M as in Theorem 3.1 be additionally strictly convex, where we also assume that $\bar{g}$ is given in geodesic polar coordinates*

$$\bar{g} = dr^2 + \vartheta^2(r) \sigma_{ij} dx^i dx^j \equiv dr^2 + \bar{g}_{ij} dx^i dx^j \tag{3.8}$$

with suitable ϑ depending on the space form. Let $p > n$. Then there exist constants c and ϵ_0 depending on n, p, $|M|$, $\|\tilde{A}\|_p$ and $\|\psi\|_\infty$, as well as a constant $\alpha = \alpha(p, n)$, such that

$$\|\mathring{A}\|_p \leq \|\tilde{H}\|_p \epsilon_0 \tag{3.9}$$

implies

$$v = \sqrt{1 + \bar{g}^{ij} u_i u_j} \le e^{\frac{cR}{\|\hat{H}\|_p^\alpha} \|\mathring{A}\|_p^\alpha}, \tag{3.10}$$

where

$$M = \{(x^0, x^i) \colon x^0 = u(x^i), (x^i) \in \mathcal{S}_0\} \tag{3.11}$$

is a suitable graph representation over a geodesic sphere $\mathcal{S}_0 \hookrightarrow N^{n+1}$ *and* $(\bar{g}^{ij})$ *is the inverse of* $(\bar{g}_{ij})$.

Proof. It is well known that a strictly convex hypersurface of $\mathbb{S}^{n+1}$ is contained in an open hemisphere, cf. [5] for the smooth case and also [12, Cor. 1.2] for the C^2-case. Thus M is covered by a conformally flat coordinate system as in Theorem 3.1, which is thus applicable. Let $\mathcal{S}_0$ be the corresponding sphere with center x_M, then we can write M as a graph over $\mathcal{S}_0$ due to the strict convexity. Thus we may apply the well-known interpolation estimate

$$v \le e^{\bar{\kappa} \operatorname{osc} u}, \tag{3.12}$$

cf. [7, Thm. 2.7.10], where

$$\operatorname{osc} u = \max u - \min u \tag{3.13}$$

and where $\bar{\kappa}$ is a lower bound for the principal curvatures of the coordinate slices $\{r = \text{const}\}$. The latter, however, only depends on $\|\psi\|_\infty$ as well. □

4. An optimality result

We prove the optimality of the estimate (1.6) in the sense that there is no hope to derive a uniform estimate of the form

$$d_{\mathcal{H}}(M, S_R(x_0)) \le c\|\mathring{A}\|_\infty^\alpha, \quad \alpha > 1, \tag{4.1}$$

in the class of uniformly C^∞-bounded hypersurfaces M. To be precise, for $\alpha > 1$ we get the following negation of (4.1) in the class of uniformly convex hypersurfaces and for all $n \ge 2$.

4.1. **Theorem.** *Let* $n \ge 2$ *and* $C = 2\max(|S_2(0)|, \|\bar{A}_{S_2}\|_\infty)$. *For all* $\alpha > 1$ *and for all* $k \in \mathbb{N}$ *there exists a uniformly convex smooth hypersurface* $M_k \hookrightarrow \mathbb{R}^{n+1}$ *with*

$$\max(\|A_k\|_\infty, |M_k|) \le C, \tag{4.2}$$

such that

$$\|\mathring{A}_k\|_\infty < \frac{1}{k} \tag{4.3}$$

and for all spheres $S \subset \mathbb{R}^{n+1}$ *there holds*

$$d_{\mathcal{H}}(M_k, S) > k\|\mathring{A}_k\|_\infty^\alpha. \tag{4.4}$$

Here $\bar{A}_{S_2}$ *denotes the second fundamental form of the sphere with radius* 2.

In a recent paper, Drach gave a counterexample to an improved spherical closeness estimate in the class of $C^{1,1}$ hypersurfaces, namely a special spindle shaped hypersurface, cf. the construction at the beginning of [6, Sec. 2] and also compare cf. [6, Thm. 1]. However, since we consider (1.4) in the space of at least C^2-hypersurfaces, we need to find a different contradiction to (4.1). This contradiction is deduced along the inverse mean curvature flow in the hyperbolic space.

Before we prove Theorem 4.1, let us for convenience recall the relevant facts about the inverse mean curvature flow in the hyperbolic space $\mathbb{H}^{n+1}$. There one considers a time parameter family of embeddings of closed, starshaped and mean-convex hypersurfaces

$$x\colon [0,T^*)\times M \hookrightarrow \mathbb{H}^{n+1}, \tag{4.5}$$

which solves

$$\dot{x} = \frac{1}{H}\nu, \tag{4.6}$$

where $H = g^{ij}h_{ij}$ and ν is the outward unit normal to $M_t = x(t,M)$. Note that we have switched the notation of H in this context due to a better comparability with the literature. It is known, cf. [8, Lemma 3.2], that for an initial starshaped and mean-convex hypersurface M_0 the flow exists for all times and all the flow hypersurfaces can be written as a graph over a fixed geodesic sphere $\mathcal{S}_0$,

$$M_t = \{(x^0,x^i)\colon x^0(t,\xi) = u(t,x^i(t,\xi))\}, \tag{4.7}$$

where u describes the radial distance to the center of $\mathcal{S}_0$. In [8, Thm. 1.2] Gerhardt claimed to have shown convergence of the rescaled hypersurfaces

$$\hat{M}_t = \text{graph } \hat{u} \equiv \text{graph}\left(u - \frac{t}{n}\right) \tag{4.8}$$

to a geodesic sphere. However, as was pointed out in [10, Thm. 1] with the help of a concrete counterexample, the limit function of $\hat{u}$ is not constant in general. In particular the authors proved that there is a starshaped and mean-convex initial hypersurface M_0, such that the limit hypersurface is not of constant curvature, in particular not a geodesic sphere. However, there is a smooth limit function to which the $\hat{M}_t$ converge smoothly, compare the proof of [8, Thm. 6.11] and also compare [16, Thm. 1.2].

In order to relate the convergence results of the IMCF in the hyperbolic space with the rigidity estimate (1.4) in the Euclidean space, we have to look at the hyperbolic flow in the conformally flat model. In [8] the Poincaré ball model in the ball of radius 2 was considered. Let r denote the geodesic distance to the center of $\mathcal{S}_0$ in $\mathbb{H}^{n+1}$, then the by the coordinate change

$$\rho = 2 - \frac{4}{e^r+1} \tag{4.9}$$

the representation of the hyperbolic metric transforms like

$$\bar{g} = dr^2 + \sinh^2(r)\sigma_{ij}dx^i dx^j = \frac{1}{\left(1-\frac{1}{4}\rho^2\right)^2}(d\rho^2 + \rho^2\sigma_{ij}dx^i dx^j) \equiv e^{2\psi}\tilde{g}, \tag{4.10}$$

where σ_{ij} is the standard round metric of the sphere $\mathcal{S}_0$. Then the convergence

$$u - \frac{t}{n} \to \hat{u}_\infty \tag{4.11}$$

in the original coordinates is equivalent to the convergence of

$$(2-w)e^{\frac{t}{n}} \to \hat{w}_\infty, \tag{4.12}$$

where

$$w = 2 - \frac{4}{e^u+1} \tag{4.13}$$

and where $\hat{w}_\infty$ is a strictly positive function due to [8, Lemma 3.1].

The proof of Theorem 4.1 is very similar to the proof of a corresponding positive result in this direction by the second author. In [17] he proved that due to a strong decay of the traceless second fundamental form along the IMCF in $\mathbb{R}^{n+1}$ we indeed obtain spherical roundness in this case without rescaling. The idea how to obtain a negative result in the hyperbolic space is that if we could improve the spherical closeness, then we could mimic the proof in [17] to deduce a roundness result in $\mathbb{H}^{n+1}$, which is not possible in view of Hung's and Wang's paper.

The idea of the proof of Theorem 4.1 goes as follows: The estimate in (4.12) provides closeness of the flow hypersurfaces to the sphere of radius 2 in the ball model. The order of the closeness is $e^{-\frac{t}{n}}$. The traceless second fundamental form decays correspondingly, as we will point in more detail later in the proof. But if we had this additional exponent α in the spherical closeness estimate, we could even deduce better spherical closeness (to a sphere different from S_2) than we have in (4.12) and then we would be able to translate this to a spherical closeness in the hyperbolic space. This would in turn yield a contradiction to Hung's and Wang's result. Now let us prove Theorem 4.1 in detail. First we need some helpful notation and an auxiliary result.

4.2. **Definition.** (i) Let N be either the Euclidean space, the hyperbolic space or an open hemisphere. For a starshaped hypersurface $M \hookrightarrow N$, let M^* be the set of points in N, with respect to which M is starshaped.

(ii) For a starshaped hypersurface $M \hookrightarrow N$ let $p \in M^*$. Then for the graph representation

$$M = \{(r, x^i) \colon r = u(x^i), (x^i) \in \mathcal{S}_p\}, \tag{4.14}$$

by

$$\operatorname{osc}_p u = \max_{x \in \mathcal{S}_p} u(x) - \min_{x \in \mathcal{S}_p} u(x) \tag{4.15}$$

we denote the oscillation of the geodesic distance of the point (u, x^i) to the point p. Here $\mathcal{S}_p$ denotes a geodesic sphere around p.

By a simple argument we obtain the following alternative for a general expanding sequence of hypersurfaces with controlled oscillation.

4.3. **Lemma.** *Let N be as in Definition 4.2 and $M_t \hookrightarrow N$, $0 \le t \in \mathbb{R}$, be a family of starshaped hypersurfaces such that*

$$M_t^* \subset M_s^* \quad \forall s \ge t \tag{4.16}$$

and such that for each $\tau_0 \ge 0$ and $p \in M_{\tau_0}^*$ there exists a constant c, such that for all $t_0 \ge \tau_0$,

$$\operatorname{osc}_p u_t \le c \operatorname{osc}_p u_{t_0} \quad \forall t \ge t_0. \tag{4.17}$$

Then for fixed p, $\operatorname{osc}_p u_t$ does not have zero as a limit value for $t \to \infty$ unless

$$\operatorname{osc}_p u_t \to 0, \quad t \to \infty. \tag{4.18}$$

Proof. For given $\epsilon > 0$, if zero is a limit point, we may choose t_0, such that

$$\operatorname{osc}_p u_{t_0} \le \frac{\epsilon}{c}, \tag{4.19}$$

then

$$\operatorname{osc}_p u_t \le c \operatorname{osc}_p u_{t_0} \le \epsilon \quad \forall t \ge t_0. \tag{4.20}$$

$\square$

Now we can prove Theorem 4.1.

Proof. Assume the contrary, i.e. that there exists $\alpha > 1$ and $k \in \mathbb{N}$, such that for all uniformly convex hypersurfaces $\tilde{M} \hookrightarrow \mathbb{R}^{n+1}$ with

$$\max(|\tilde{M}|, \|\tilde{A}\|_\infty) \leq C \tag{4.21}$$

we have that

$$\|\mathring{\tilde{A}}\|_\infty < \frac{1}{k} \tag{4.22}$$

implies

$$\tilde{d}_{\mathcal{H}}(\tilde{M}, \tilde{S}) \leq k\|\mathring{\tilde{A}}\|_\infty^\alpha \tag{4.23}$$

for some suitable sphere $\tilde{S} \subset \mathbb{R}^{n+1}$, where the Hausdorff distance is measured with respect to the Euclidean metric. According to [10, Thm. 1] for $n = 2$ and [10, Sec. 4] for $n \geq 3$ there exists a starshaped and mean-convex hypersurface $M_0 \hookrightarrow \mathbb{H}^{n+1}$, such that for no graph representation

$$M_t = \text{graph } u \tag{4.24}$$

the rescaled IMCF flow hypersurfaces

$$\hat{M}_t = \text{graph}\left(u - \frac{t}{n}\right) \equiv \text{graph } \hat{u} \tag{4.25}$$

converge to a geodesic sphere. However, for each graph representation, we obtain smooth convergence of

$$\hat{u} \to \hat{u}_\infty. \tag{4.26}$$

In [16, Thm. 1.2 (2)] it is deduced that

$$\|\mathring{A}\|_\infty \leq ce^{-\frac{2t}{n}}, \tag{4.27}$$

where $c = c(n, M_0)$. Now fix a graph representation around $p \in M_0^*$. From (3.6) we obtain that the corresponding Euclidean traceless part decays like

$$\|\mathring{\tilde{A}}\|_\infty = \|e^\psi \mathring{A}\|_\infty \leq e^\psi_{\max} e^{-\frac{2t}{n}}, \tag{4.28}$$

where

$$e^\psi_{\max} = \frac{1}{\left(1 - \frac{1}{4}w^2_{\max}\right)} \tag{4.29}$$

with w as in (4.13) and

$$w_{\max} = \max_{x \in \mathcal{S}_p} w(x). \tag{4.30}$$

Due to (4.12) we obtain

$$\|\mathring{\tilde{A}}\|_\infty \leq ce^{-\frac{t}{n}} \tag{4.31}$$

and due to the C^∞-convergence of $w \to 2$, we are in the situation to apply our assumption and obtain (4.23), whenever t is large enough. We obtain a sequence of spheres $\tilde{S}_{\tilde{R}_t} \subset \mathbb{R}^{n+1}$, such that

$$\tilde{d}_{\mathcal{H}}(\tilde{M}_t, \tilde{S}_{\tilde{R}_t}) \leq ce^{-\frac{\alpha}{n}t}. \tag{4.32}$$

Due to (4.12) we even have

$$\tilde{S}_{\tilde{R}_t} \subset B_2(0), \tag{4.33}$$

for large times t.

Now let us switch back to the hyperbolic space. The spheres $\tilde{S}_{\tilde{R}_t}$ are geodesic spheres in $\mathbb{H}^{n+1}$ as well since total umbilicity is preserved under a conformal transformation and in the Euclidean space as well as in the hyperbolic space for closed and embedded hypersurfaces total umbilicity is tantamount to being a geodesic sphere. We denote these spheres in $\mathbb{H}^{n+1}$ by S_{R_t}. For the corresponding hyperbolic Hausdorff distance we deduce

$$d_{\mathcal{H}}(M_t, S_{R_t}) \le e^{\psi}_{\max} \tilde{d}_{\mathcal{H}}(\tilde{M}_t, \tilde{S}_{\tilde{R}_t}) \le ce^{\frac{1-\alpha}{n}t}, \tag{4.34}$$

which converges to 0 as $t \to \infty$.

Since the inradius of the M_t converges to infinity and for large t the M_t are strictly convex, for each $\delta > 0$ we find $t_0 > 0$, such that

$$\bar{B}_\delta(p) \subset M^*_{t_0} \subset M^*_t \quad \forall t \ge t_0, \tag{4.35}$$

where the latter inclusion is due to the fact that starshapedness around a given point is preserved. According to [16, Prop. 3.2, Lemma 3.5], there holds for the oscillation of u that for all τ_0, all $q \in M^*_{\tau_0}$ and all $t_0 \ge \tau_0$ we have

$$\operatorname{osc}_q u(t, \cdot) \le c \operatorname{osc}_q u(t_0, \cdot) \quad \forall t \ge t_0, \tag{4.36}$$

where c depends on n and on a lower bound on the minimal distance of q to M_{τ_0}. So in particular, if we choose

$$\delta = c \operatorname{osc}_p u(0, \cdot), \tag{4.37}$$

we find that the oscillation of each M_t is minimized within the set $\bar{B}_\delta(p)$:

$$\operatorname*{argmin}_{q \in M^*_t} \operatorname{osc}_q u(t, \cdot) \in \bar{B}_\delta(p) \quad \forall t \ge t_0, \tag{4.38}$$

because outside $\bar{B}_\delta(p)$ the oscillation is already larger than it is with respect to p.

Due to (4.34) we obtain

$$\operatorname{osc}_{q_t} u(t, \cdot) = \min_{q \in \bar{B}_\delta(p)} \operatorname{osc}_q u(t, \cdot) \le ce^{\frac{1-\alpha}{n}t} \quad \forall t \ge t_0. \tag{4.39}$$

Let t_k be a sequence of times with $t_k \to \infty$. Due to the compactness of $\bar{B}_\delta(p)$ a subsequence of center points converges,

$$q_{t_k} \equiv q_k \to q \in \bar{B}_\delta(p), \tag{4.40}$$

where we did not rename the index of the sequence. Since

$$|\operatorname{osc}_{q_k} u(t_k, \cdot) - \operatorname{osc}_q u(t_k, \cdot)| \le 2 \operatorname{dist}(q_k, q) \quad \forall k \in \mathbb{N}, \tag{4.41}$$

we obtain in view of (4.39),

$$\operatorname{osc}_q u(t_k, \cdot) \to 0, \quad k \to \infty. \tag{4.42}$$

In view of (4.36) and the preservation of starshapedness along IMCF the assumptions of Lemma 4.3 are fulfilled. Applying Lemma 4.3, we obtain that

$$\operatorname{osc}_q u(t, \cdot) \to 0, \tag{4.43}$$

in contradiction to the choice of the initial hypersurface. □

4.4. *Remark.* Note that in turn of the proof we even have shown that for given $\alpha > 1$ and $k \in \mathbb{N}$ as in Theorem 4.1, such a counterexample M_k satisfying (4.3) and (4.4) must actually occur along the inverse mean curvature flow in the conformally flat version of the IMCF in $\mathbb{H}^{n+1}$. We only used our contrary assumption within this class of flow hypersurfaces.

5. Concluding remark

We would like to point out that the techniques in Theorem 4 might be useful in other situations. Whenever one would like to estimate the closeness to a sphere in comparison with another geometric quantity, e.g. in comparison with eigenvalue pinching of the Laplacian or also in almost-Schur/almost-CMC type estimates, one could determine how this particular geometric quantity behaves along the IMCF and then determine the best possible roundness estimate using the IMCF in $\mathbb{H}^{n+1}$. It should often be quite straightforward to derive the best possible decay estimate.

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APPENDIX A3. RIGIDITY OF ALMOST-UMBILICAL HYPERSURFACES

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Appendix A4

INVERSE CURVATURE FLOWS IN RIEMANNIAN WARPED PRODUCTS

by Julian Scheuer
submitted for publication

INVERSE CURVATURE FLOWS IN RIEMANNIAN WARPED PRODUCTS

JULIAN SCHEUER

ABSTRACT. The long-time existence and umbilicity estimates for compact, graphical solutions to expanding curvature flows are deduced in Riemannian warped products of a real interval with a compact fibre. Notably we do not assume the ambient manifold to be rotationally symmetric, nor the radial curvature to converge, nor a lower bound on the ambient sectional curvature. The inverse speeds are given by powers $p \leq 1$ of a curvature function satisfying few common properties.

1. Introduction

This paper deals with expanding curvature flows of the form

$$\dot{x} = \frac{1}{F^p}\nu, \quad 0 < p \leq 1, \tag{1.1}$$

where

$$x\colon [0, T^*) \times M^n \to N^{n+1}, \quad n \geq 2,$$

is a family of embeddings of a smooth, orientable, compact manifold M^n and $N = N^{n+1}$ is a product

$$N = (R_0, \infty) \times \mathcal{S}_0$$

with metric

$$\bar{g} = dr^2 + \vartheta^2(r)\sigma.$$

Here $\vartheta \in C^\infty((R_0, \infty))$ satisfies $\vartheta' > 0$, $\vartheta'' \geq 0$ and $(\mathcal{S}_0, \sigma)$ is a compact Riemannian manifold. In (1.1), F is a function evaluated at the Weingarten operator $\mathcal{W}$ of the flow hypersurfaces $M_t = x(t, M)$ at the respective point x and ν is the outward pointing normal, i.e.

$$\bar{g}(\nu, \partial_r) > 0.$$

The detailed assumptions on the curvature function F and on N are the following.

1.1. **Assumption.** Let $\Gamma \subset \mathbb{R}^n$ be an open, symmetric and convex cone containing the positive cone

$$\Gamma_+ = \{(\kappa_i) \in \mathbb{R}^n \colon \kappa_i > 0 \quad \forall 1 \leq i \leq n\}$$

and suppose $f \in C^\infty(\Gamma)$ is a positive, symmetric, strictly monotone, 1-homogeneous and concave function with

$$f(1, \ldots, 1) = n, \quad f_{|\partial\Gamma} = 0$$

and associated curvature function $F = F(\mathcal{W})$, cf. section 2.2.

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Particular examples of curvature functions satisfying these assumptions are roots or quotients of elementary symmetric polynomials,

$$F = nH_k^{\frac{1}{k}}, \quad F = n\frac{H_{k+1}}{H_k}$$

and many more, cf. [4].

In order to obtain good asymptotics we will make the following assumption on the warping function. This assumption will not be needed for the long-time existence.

1.2. **Assumption.** Assume the warping function $\vartheta \in C^\infty((R_0,\infty))$ to satisfy

$$\limsup_{r\to\infty} \frac{\vartheta''\vartheta}{\vartheta'^2} < \infty \quad \text{and} \quad \limsup_{\substack{r\to\infty \\ \vartheta''(r)>0}} \frac{\vartheta'''\vartheta}{\vartheta'\vartheta''} < \infty.$$

In the following theorem $\underline{\widehat{\mathrm{Rc}}}$ denotes the smallest eigenvalue of the Ricci tensor of σ and H_k denotes the curvature function determined by the k-th normalized elementary symmetric polynomial of the principal curvatures, compare section 2.2 for further information. In this paper we aim to prove the following theorem.

1.3. **Theorem.** *Let $(\mathcal{S}_0,\sigma)$ be a smooth, compact and orientable Riemannian manifold of dimension $n \geq 2$, $R_0 > 0$, $N = (R_0,\infty)\times\mathcal{S}_0$ and define a warped product metric on N,*

$$\bar{g} = dr^2 + \vartheta^2(r)\sigma,$$

with $\vartheta \in C^\infty((R_0,\infty))$, $\vartheta'' \geq 0$ and $\vartheta' > 0$. Let $0 < p \leq 1$ and F satisfy Assumption 1.1. Let

$$x_0 \colon M \hookrightarrow N$$

be the embedding of a hypersurface M_0, which is graphical over $\mathcal{S}_0$, i.e. there exists $u \in C^\infty(\mathcal{S}_0,(R_0,\infty))$ such that

$$M_0 = \{(u(y),y) \colon y \in \mathcal{S}_0\},$$

and such that all its n-tuples of principal curvatures belong to Γ.

(i) Assume either of the following properties to hold:

$$(a)\ \sigma \text{ has non-negative sectional curvature.}$$

$$(b)\ F = n\frac{H_{k+1}}{H_k}, \quad 0 \leq k \leq n-1.$$

Then there exists a unique immortal solution

$$x \colon [0,\infty)\times M \to N$$

of

$$\begin{aligned} \dot{x} &= \frac{1}{F^p}\nu \\ x(0,\cdot) &= x_0, \end{aligned} \tag{1.2}$$

which is also graphical over $\mathcal{S}_0$, i.e. $\langle \nu, \partial_r\rangle > 0$.

(ii) Assume σ has non-negative sectional curvature and each of the following properties:

(A) Assumption 1.2 holds.

$$(B)\ \sup_{r>0}\vartheta'(r) < \infty \text{ and } p = 1 \quad \Rightarrow \quad \underline{\widehat{\mathrm{Rc}}} > 0 \text{ and } F = n\frac{H_{k+1}}{H_k}, \quad 0 \leq k \leq n-1.$$

$$(C)\ \sup_{r>0}\vartheta'(r) = \infty \text{ and } p = 1 \quad \Rightarrow \quad \liminf_{r\to\infty}\frac{\vartheta''\vartheta}{\vartheta'^2} > 0.$$

Then the flow hypersurfaces become umbilical with the rate

$$\left|h^i_j - \frac{\vartheta'}{\vartheta}\delta^i_j\right| \leq ct\frac{\vartheta'^{1-p(p+1)}}{\vartheta}, \tag{1.3}$$

where the t-factor may be dropped in case $p < 1$ or bounded ϑ' and may even be replaced by $e^{-\alpha t}$ for some positive α if ϑ' is bounded and $p = 1$.

Let us make some remarks on the technical assumptions made in Theorem 1.3.

1.4. *Remark.* (i) The assumptions in statement (i) of Theorem 1.3 are optimal in the sense, that for example in a spherical ambient space with $\vartheta'' < 0$ the inverse mean curvature flow only exists for a finite time, cf. [26, 51] and for $p > 1$ the maximal existence is finite if $N = \mathbb{R}^{n+1}$, cf. [25].

(ii) The assumption on the sectional curvature of σ can be relaxed. The crucial point, where we use this assumption is in the first gradient estimates, especially in estimate (3.3), where we throw away the term involving $\widehat{\mathrm{Rm}}$, if F is general. However, under a further suitable technical assumption we could also absorb it into the first line of this equation. For the special case of the inverse mean curvature flow in the Reissner-Nordström manifolds this has been accomplished in the recent preprint [9]. However, in order to avoid too many technical assumptions, we will not improve the main result in this direction here, except that we prove the long-time existence in general, provided that F is a quotient of the H_k. For the IMCF this was also accomplished in [43, 71].

(iii) The rates of convergence in this theorem can be improved, if the ambient sectional curvatures approach each other at infinity. Such results have been accomplished for example in [10, 49, 60] in case $p = 1$ and in [58] in case $p < 1$ in the hyperbolic space. Since the main aim of this work is to deal with spaces in which the limits of the quantities in Assumption 1.2 do not exist (if σ is the round metric this implies that N is not asymptotically a spaceform), we will not pursue these optimal estimates here and stick to the best we could accomplish in general ambient spaces. To the best of my knowledge, the only result in such general spaces is the analogous result for the inverse mean curvature flow proven in [60].

(iv) The question, whether (1.3) implies that the flow hypersurfaces do become almost umbilical, depends on the ambient space N and on p. However, if $p = 1$, the analysis in [60, Prop. 3.1] implies that ϑ grows exponentially. Hence in this case we obtain exponential decay of $\mathcal{W} - \frac{\vartheta'}{\vartheta}\,\mathrm{id}$.

(v) In case $p = 1$, the gradient decay estimates obtained in Lemma 4.8 are optimal even if the ambient space is asymptotically a spaceform. Compare the explanation in [60, Rem. 1.5].

(vi) In case $p = 1$ the estimate (1.3) turned out to be strong enough to obtain geometric inequalities, for example in [5, 21, 50, 67]. We are optimistic that Theorem 1.3 will be helpful with such applications as well.

The motivation to analyse the behaviour of inverse curvature flows has mostly been driven by their power to deduce geometric inequalities for hypersurfaces. The most prominent example is the proof of the Riemannian Penrose inequality due to Huisken/Ilmanen [33], building on the observation made by Geroch [27] and Jang/Wald [38] that the Hawking mass of a connected surface is non-decreasing under the inverse mean curvature flow (IMCF) with $F = H$ and $p = 1$, if the ambient scalar curvature is non-negative. Since for general initial data the IMCF may develop singularities, Huisken and Ilmanen defined a notion of a weak solution for this flow, maintaining the Geroch monotonicity. This enabled them to prove the Riemannian Penrose inequality. For a short outline of their procedure also compare [32].

Also the classical solution to IMCF has lead to very interesting applications. A crucial feature of this flow in $\mathbb{R}^{n+1}$ is that one does not need to require convexity of the initial hypersurface to avoid finite time singularities. Namely, Gerhardt [22] and Urbas [65] proved the long-time existence even for more general flows in $\mathbb{R}^{n+1}$, with F satisfying Assumption 1.1, $p = 1$ and a starshaped initial hypersurface M_0 with $F_{|M_0} > 0$. Furthermore, after exponential rescaling, the flow converges to a sphere smoothly. This result, with $F = nH_{k+1}/H_k$, was later exploited by Guan/Li [29] to generalise the Alexandrov-Fenchel quermassintegral inequalities from the convex setting to the starshaped and H_{k+1}-convex setting. Since then a cascade of similar results followed by the same method (monotone quantity plus some convergence result) in various ambient spaces. The tough parts are to find the monotone quantity and to prove a sufficient convergence result. Examples of other results in this direction are a generalised Minkowski-type inequality in the anti-de Sitter-Schwarzschild manifold due to Brendle/Hung/Wang [5], Alexandrov-Fenchel-type inequalities in the hyperbolic space [16, 20, 31, 44, 69] and in the sphere [28, 51, 69]. Further similar applications can be found in [21, 41, 50, 67].

In many of these papers, there was a need to investigate the asymptotical behaviour of the corresponding inverse curvature flow separately, since a unified treatment had not been present. Hence, a branch of research solely dealing with inverse curvature flows has developed within the community, where the main aims are to generalise the convergence results in various directions (concerning flow speed and ambient space). A step towards generalising the ambient space was made by the author with the paper [60], where the IMCF was considered in rotationally symmetric warped products under assumptions similar to Assumption 1.2. Before (and after) that, some more special ambient spaces were treated, which, to the best of my knowledge, all assumed convergence of the quantities in Assumption 1.2. Instead of giving a description of the available results verbally, the following table is supposed to give an overview as broad as I could accomplish over the previous results on *smooth, inverse curvature flows of closed hypersurfaces in Riemannian warped products.* The topics they cover are for example long-time existence, asymptotic behaviour, solitons and others. We point out that, in order to keep things manageable, we leave aside treatments of contracting flows, weak solutions, flows in Lorentzian manifolds, flows of entire graphs, flows with boundary conditions, anisotropic flows and flows with constraints (e.g. volume preserving flows).

N/F		$F = n\frac{H_{k+1}}{H_k}$	F more general and $p = 1$	$p \neq 1$ or non-hom. speed
CSC	$\mathbb{R}^{n+1}$	[8, 11, 18] [34]	[15, 22, 45] [64, 65, 66]	[2, 3, 7, 12, 13] [14, 25, 36, 37] [40, 42, 46, 47] [59, 63, 68]
	$\mathbb{H}^{n+1}$	[17, 35]	[24, 48, 70]	[42, 57, 58, 68]
	$\mathbb{S}^{n+1}$		[26, 48]	[6, 7, 51, 68]
Asympt. CSC	$\mathbb{R}^{n+1}$	[17, 43, 50]		
	$\mathbb{H}^{n+1}$	[5, 49, 53]	[10]	
More general	$\frac{\vartheta''\vartheta}{\vartheta'^2}$ converges	[9, 43, 52, 71]		
	1.2	[60]		

Note that a reference only appears in the most general slot it can be placed. Also note that there are few works on the inverse mean curvature flow in ambient spaces which are not warped products, [1, 39, 55, 56]. This paper aims to fill some gaps in this table, especially in the two bottom rows, and is organised as follows. Section 2 collects some notation, conventions, basic facts about curvature functions and the relevant evolution equations. In section 3 we treat the long-time existence and in section 4 we analyse the asymptotic behaviour and finish the proof of Theorem 1.3.

2. Preliminaries

2.1. **Notation and conventions.** In this paper we deal with embedded hypersurfaces

$$x\colon M \hookrightarrow N$$

of a smooth, closed and orientable manifold M^n into an ambient Riemannian manifold $(N^{n+1},\bar{g})$. All geometric quantities of N will be furnished with an overbar, e.g. $\bar{g}=(\bar{g}_{\alpha\beta})$ for the metric, $\bar{\nabla}$ for its Levi-Civita connection etc. In coordinate expressions, greek indices run from 0 to n. For the quantities induced by the embedding x, we use latin indices running from 1 to n, e.g. for the induced metric $g=(g_{ij})$ with Levi-Civita connection ∇. For a (k,l) tensor field T on M, its covariant derivative ∇T is a $(k,l+1)$ tensor field given by

$$\begin{aligned}
&(\nabla T)(Y^1,\dots,Y^k,X_1,\dots,X_l,X)\\
&=(\nabla_X T)(Y^1,\dots,Y^k,X_1,\dots,X_l)\\
&=X(T(Y^1,\dots,Y^k,X_1,\dots,X_l))-T(\nabla_X Y^1,Y^2,\dots,Y^k,X_1,\dots,X_l)-\dots\\
&\quad -T(Y^1,\dots,Y^k,X_1,\dots,X_{l-1}\nabla_X X_l),
\end{aligned}$$

the coordinate expression of which is denoted by

$$\nabla T=\left(T^{i_1\dots i_k}_{j_1\dots j_l;j_{l+1}}\right).$$

The index appearing after the semicolon indicates the derivative index.

Our convention for the $(1,3)$-Riemannian curvature tensor Rm of a connection ∇ is

$$\mathrm{Rm}(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z,$$

where X,Y,Z are vector fields and where $[X,Y]$ is the Lie-bracket

$$[X,Y]\varphi=X(Y\varphi)-Y(X\varphi)\quad \forall \varphi\in C^\infty(M).$$

The purely covariant Riemannian curvature tensor is defined by lowering to the fourth slot:

$$\mathrm{Rm}(X,Y,Z,W)=g(\mathrm{Rm}(X,Y)Z,W).$$

Finally the Ricci curvature is

$$\mathrm{Rc}(X,Y)=\mathrm{tr}\left(\mathrm{Rm}(\cdot,X)Y\right).$$

For metrics (g_{ij}) we always denote its dual by (g^{ij}), i.e.

$$\delta^i_j=g^{ik}g_{kj}.$$

The induced geometry of M is governed by the following relations. The second fundamental form $h=(h_{ij})$ is defined by the Gaussian formula

$$\bar{\nabla}_X Y=\nabla_X Y-h(X,Y)\nu, \tag{2.1}$$

where ν is a normal field. The Weingarten endomorphism $\mathcal{W}=(h^i_j)$ is defined by $h^i_j=g^{ki}h_{kj}$ and we have the Weingarten equation

$$\bar{\nabla}_X\nu=\mathcal{W}(X). \tag{2.2}$$

We also have the Codazzi equation

$$\nabla_Z h(X,Y)-\nabla_Y h(X,Z)=-\overline{\mathrm{Rm}}(\nu,X,Y,Z).$$

Let us record this equation is coordinates:

$$h_{ij;k}-h_{ik;j}=-\overline{\mathrm{Rm}}(\nu,x_{;i},x_{;j},x_{;k}). \tag{2.3}$$

The Gauss equation states

$$\mathrm{Rm}(W,X,Y,Z)=\overline{\mathrm{Rm}}(W,X,Y,Z)+h(W,Z)h(X,Y)-h(W,Y)h(X,Z) \tag{2.4}$$

or in coordinates

$$R_{ijkl}=\overline{\mathrm{Rm}}(x_{;i},x_{;j},x_{;k},x_{;l})+h_{il}h_{jk}-h_{ik}h_{jl}.$$

Warped products. Throughout this paper we assume that the ambient manifold is a *warped product* of the form

$$(N,\bar{g})=(I\times\mathcal{S}_0,\bar{g}),$$

where $I=(R_0,\infty)$, $(\mathcal{S}_0,\sigma)$ is an n-dimensional compact Riemannian manifold and

$$\bar{g}=dr^2+\vartheta^2(r)\sigma \tag{2.5}$$

with $\vartheta\in C^\infty((R_0,\infty))$. We will need to know how the curvature tensor of $\bar{g}$ arises from the curvature tensors of dr^2 and σ. The relevant formulae can be found in [54, Ch. 7, Prop. 42]. We state them here for further use, but adapted to our curvature convention, which differs from the one in op. cit. We denote by $\mathscr{L}(\mathbb{R})$ and $\mathscr{L}(\mathcal{S}_0)$ the space of all vector field on $\mathbb{R}$ resp. $\mathcal{S}_0$ lifted to N.

2.1. **Lemma.** ([54, Ch. 7, Prop. 42]) *Let N be given as above. If $X,Y,Z\in\mathscr{L}(\mathbb{R})$ and $U,V,W\in\mathscr{L}(\mathcal{S}_0)$, then the Riemannian curvature tensor of N is given by*

(i) $\overline{\mathrm{Rm}}(X,Y)Z=0$,
(ii) $\overline{\mathrm{Rm}}(V,X)Y=-\frac{\bar{\nabla}^2\vartheta(X,Y)}{\vartheta}V=-\frac{\vartheta''}{\vartheta}\bar{g}(X,Y)V$,
(iii) $\overline{\mathrm{Rm}}(X,Y)V=\overline{\mathrm{Rm}}(V,W)X=0$,
(iv) $\overline{\mathrm{Rm}}(X,V)W=-\frac{\vartheta''}{\vartheta}\bar{g}(V,W)X$
(v) $\overline{\mathrm{Rm}}(V,W)U=\widetilde{\mathrm{Rm}}(V,W)U-\frac{\vartheta'^2}{\vartheta^2}(\bar{g}(W,U)V-\bar{g}(V,U)W)$,

where $\widetilde{\mathrm{Rm}}$ is the lift of the Riemann tensor of the fibre $(\mathcal{S}_0,\vartheta^2(r)\sigma)$ under the projection $\pi\colon N\to\mathcal{S}_0$.

It will turn out to be convenient to have a closed coordinate expression for $\overline{\mathrm{Rm}}$, which follows easily from checking all of the five cases.

2.2. **Lemma.** *In coordinates the Riemannian curvature tensor of the warped product*

$$(N,\bar{g})=(I\times\mathcal{S}_0,dr^2+\vartheta^2(r)\sigma)$$

is given by

$$\bar{R}^{\epsilon}_{\alpha\beta\gamma}=\left(\left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)\bar{S}_{\alpha'\beta'\gamma'\delta'}+\widetilde{R}_{\alpha'\beta'\gamma'\delta'}\right)P^{\alpha'}_{\alpha}P^{\beta'}_{\beta}P^{\gamma'}_{\gamma}P^{\delta'\epsilon}-\frac{\vartheta''}{\vartheta}\bar{S}^{\epsilon}_{\alpha\beta\gamma}, \tag{2.6}$$

where

$$\bar{S}^{\epsilon}_{\alpha\beta\gamma}=\bar{g}_{\beta\gamma}\delta^{\epsilon}_{\alpha}-\bar{g}_{\alpha\gamma}\delta^{\epsilon}_{\beta}$$

and

$$P = \mathrm{id} - \frac{\partial}{\partial r} \otimes dr.$$

Hence we obtain a formula for the derivative of $\overline{\mathrm{Rm}}$.

2.3. **Lemma.** *The coordinate functions of the covariant derivative of the* $(0,4)$*-curvature tensor are given by*

$$\begin{aligned}
\bar{R}_{\alpha\beta\gamma\delta;\epsilon} = &-\left(\frac{\vartheta''}{\vartheta}\right)' r_{;\epsilon}\bar{S}_{\alpha\beta\gamma\delta} + \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)' r_{;\epsilon}\bar{S}_{\alpha'\beta'\gamma'\delta'}P^{\alpha'}_{\alpha}P^{\beta'}_{\beta}P^{\gamma'}_{\gamma}P^{\delta'}_{\delta} \\
&+ \widetilde{R}_{\alpha'\beta'\gamma'\delta';\epsilon}P^{\alpha'}_{\alpha}P^{\beta'}_{\beta}P^{\gamma'}_{\gamma}P^{\delta'}_{\delta} - \frac{\vartheta'}{\vartheta}r_{;\alpha}\bar{T}_{\alpha'\beta'\gamma'\delta'}P^{\alpha'}_{\epsilon}P^{\beta'}_{\beta}P^{\gamma'}_{\gamma}P^{\delta'}_{\delta} \\
&- \frac{\vartheta'}{\vartheta}r_{;\beta}\bar{T}_{\alpha'\beta'\gamma'\delta'}P^{\alpha'}_{\alpha}P^{\beta'}_{\epsilon}P^{\gamma'}_{\gamma}P^{\delta'}_{\delta} - \frac{\vartheta'}{\vartheta}r_{;\gamma}\bar{T}_{\alpha'\beta'\gamma'\delta'}P^{\alpha'}_{\alpha}P^{\beta'}_{\beta}P^{\gamma'}_{\epsilon}P^{\delta'}_{\delta} \\
&- \frac{\vartheta'}{\vartheta}r_{;\delta}\bar{T}_{\alpha'\beta'\gamma'\delta'}P^{\alpha'}_{\alpha}P^{\beta'}_{\beta}P^{\gamma'}_{\gamma}P^{\delta'}_{\epsilon},
\end{aligned} \tag{2.7}$$

where

$$\bar{T}_{\alpha'\beta'\gamma'\delta'} = \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)\bar{S}_{\alpha'\beta'\gamma'\delta'} + \tilde{R}_{\alpha'\beta'\gamma'\delta'}.$$

Proof. Denote by $\bar{\Gamma}^{\gamma}_{\alpha\beta}$ the Christoffel symbols, i.e.

$$\bar{\Gamma}^{\gamma}_{\alpha\beta}\frac{\partial}{\partial x^{\gamma}} = \bar{\nabla}_{\frac{\partial}{\partial x^{\alpha}}}\frac{\partial}{\partial x^{\beta}}$$

and

$$\bar{\Gamma}^{\gamma}_{\alpha\beta} = \frac{1}{2}\bar{g}^{\gamma\delta}\left(\frac{\partial}{\partial x^{\beta}}\bar{g}_{\alpha\delta} + \frac{\partial}{\partial x^{\alpha}}\bar{g}_{\beta\delta} - \frac{\partial}{\partial x^{\delta}}\bar{g}_{\alpha\beta}\right).$$

Using the definition of the metric we see

$$\bar{\Gamma}^{0}_{\alpha\epsilon} = -\frac{\vartheta'}{\vartheta}\bar{g}_{\alpha'\beta'}P^{\alpha'}_{\alpha}P^{\beta'}_{\epsilon} = -\frac{\vartheta'}{\vartheta}\bar{g}_{\alpha'\epsilon}P^{\alpha'}_{\alpha}, \quad \bar{\Gamma}^{\alpha'}_{0\epsilon} = \frac{\vartheta'}{\vartheta}P^{\alpha'}_{\epsilon}$$

and hence there holds

$$\begin{aligned}
P^{\alpha'}_{\alpha;\epsilon} = -r^{\alpha'}_{;\epsilon}r_{\alpha} - r_{;}{}^{\alpha'}r_{;\alpha\epsilon} &= -\bar{\Gamma}^{\alpha'}_{0\epsilon}r_{;\alpha} + r_{;}{}^{\alpha'}\bar{\Gamma}^{0}_{\alpha\epsilon} \\
&= -\frac{\vartheta'}{\vartheta}P^{\alpha'}_{\epsilon}r_{;\alpha} - \frac{\vartheta'}{\vartheta}r_{;}{}^{\alpha'}\bar{g}_{\gamma'\epsilon}P^{\gamma'}_{\alpha}.
\end{aligned}$$

There holds

$$\bar{T}_{\alpha'\beta'\gamma'\delta'}r_{;}{}^{\alpha'}P_{\epsilon\alpha}P^{\beta'}_{\beta}P^{\gamma'}_{\gamma}P^{\delta'}_{\delta} = 0$$

and hence differentiation of (2.6) gives

$$\begin{aligned}
\bar{R}_{\alpha\beta\gamma\delta;\epsilon} = &-\left(\frac{\vartheta''}{\vartheta}\right)' r_{;\epsilon}\bar{S}_{\alpha\beta\gamma\delta} + \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)' r_{;\epsilon}\bar{S}_{\alpha'\beta'\gamma'\delta'}P^{\alpha'}_{\alpha}P^{\beta'}_{\beta}P^{\gamma'}_{\gamma}P^{\delta'}_{\delta} \\
&+ \tilde{R}_{\alpha'\beta'\gamma'\delta';\epsilon}P^{\alpha'}_{\alpha}P^{\beta'}_{\beta}P^{\gamma'}_{\gamma}P^{\delta'}_{\delta} - \frac{\vartheta'}{\vartheta}r_{;\alpha}\bar{T}_{\alpha'\beta'\gamma'\delta'}P^{\alpha'}_{\epsilon}P^{\beta'}_{\beta}P^{\gamma'}_{\gamma}P^{\delta'}_{\delta} \\
&- \frac{\vartheta'}{\vartheta}r_{;\beta}\bar{T}_{\alpha'\beta'\gamma'\delta'}P^{\alpha'}_{\alpha}P^{\beta'}_{\epsilon}P^{\gamma'}_{\gamma}P^{\delta'}_{\delta} - \frac{\vartheta'}{\vartheta}r_{;\gamma}\bar{T}_{\alpha'\beta'\gamma'\delta'}P^{\alpha'}_{\alpha}P^{\beta'}_{\beta}P^{\gamma'}_{\epsilon}P^{\delta'}_{\delta} \\
&- \frac{\vartheta'}{\vartheta}r_{;\delta}\bar{T}_{\alpha'\beta'\gamma'\delta'}P^{\alpha'}_{\alpha}P^{\beta'}_{\beta}P^{\gamma'}_{\gamma}P^{\delta'}_{\epsilon},
\end{aligned}$$

which is the claimed formula. □

We will later have to deal with the $\tilde{R}_{\alpha'\beta'\gamma'\delta';\epsilon}$-term in (2.7).

2.4. **Lemma.** *For every $r_0 > R_0$ there exists a constant c such that*

$$\|\bar\nabla\widetilde{\mathrm{Rm}}\| \le c\frac{\vartheta'}{\vartheta^3}.$$

Proof. We define a $\bar g$-orthonormal frame $(\tilde e_\alpha)_{0\le\alpha\le n}$ as follows:

$$e_0 = \tilde e_0 = \partial_r$$

and, given a σ-orthonormal frame $(e_i)_{1\le i\le n}$ on $\mathcal{S}_0$ we put

$$\tilde e_i = \vartheta^{-1}e_i.$$

Then clearly

$$\bar g(\tilde e_\alpha, \tilde e_\beta) = \delta_{\alpha\beta}, \quad 0\le\alpha,\beta\le n.$$

To prove the lemma, it suffices to estimate the components of $\bar\nabla\widetilde{\mathrm{Rm}}$ accordingly with respect to this frame. There holds

$$\begin{aligned}\bar\nabla_{\tilde e_\epsilon}\widetilde{\mathrm{Rm}}(\tilde e_\alpha,\tilde e_\beta,\tilde e_\gamma,\tilde e_\delta) = \tilde e_\epsilon\left(\widetilde{\mathrm{Rm}}(\tilde e_\alpha,\tilde e_\beta,\tilde e_\gamma,\tilde e_\delta)\right) &- \widetilde{\mathrm{Rm}}(\bar\nabla_{\tilde e_\epsilon}\tilde e_\alpha,\tilde e_\beta,\tilde e_\gamma,\tilde e_\delta)\\ &- \widetilde{\mathrm{Rm}}(\tilde e_\alpha,\bar\nabla_{\tilde e_\epsilon}\tilde e_\beta,\tilde e_\gamma,\tilde e_\delta) - \widetilde{\mathrm{Rm}}(\tilde e_\alpha,\tilde e_\beta,\bar\nabla_{\tilde e_\epsilon}\tilde e_\gamma,\tilde e_\delta)\\ &- \widetilde{\mathrm{Rm}}(\tilde e_\alpha,\tilde e_\beta,\tilde e_\gamma,\bar\nabla_{\tilde e_\epsilon}\tilde e_\delta).\end{aligned}\tag{2.8}$$

There holds

$$\widetilde{\mathrm{Rm}}(\tilde e_\alpha,\tilde e_\beta,\tilde e_\gamma,\tilde e_\delta) = \vartheta^{-2}\widehat{\mathrm{Rm}}(\pi_*e_\alpha,\pi_*e_\beta,\pi_*e_\gamma,\pi_*e_\delta),$$

where $\widehat{\mathrm{Rm}}$ is the Riemann tensor of σ. Hence

$$\tilde e_\epsilon\left(\widetilde{\mathrm{Rm}}(\tilde e_\alpha,\tilde e_\beta,\tilde e_\gamma,\tilde e_\delta)\right) = \begin{cases}-\frac{2\vartheta'}{\vartheta^3}\widehat{\mathrm{Rm}}(\pi_*e_\alpha,\pi_*e_\beta,\pi_*e_\gamma,\pi_*e_\delta), & \epsilon = 0\\ \vartheta^{-3}e_\epsilon\left(\widehat{\mathrm{Rm}}(\pi_*e_\alpha,\pi_*e_\beta,\pi_*e_\gamma,\pi_*e_\delta)\right), & \epsilon\neq 0.\end{cases}\tag{2.9}$$

From [54, Ch. 7, Prop. 35] we obtain

$$\pi_*\bar\nabla_{\tilde e_\epsilon}\tilde e_\alpha = \begin{cases}\frac{\vartheta'}{\vartheta}\pi_*\tilde e_\alpha, & \epsilon = 0\\ \hat\nabla_{\tilde e_\epsilon}\tilde e_\alpha, & \epsilon\neq 0, \alpha\neq 0\\ \frac{\vartheta'}{\vartheta}\tilde e_\epsilon, & \epsilon\neq 0, \alpha = 0,\end{cases}\tag{2.10}$$

where $\hat\nabla$ is the Levi-Civita connection of σ. In case $\epsilon\neq 0$, $\alpha\neq 0$ we have

$$\hat\nabla_{\tilde e_\epsilon}\tilde e_\alpha = \vartheta^{-1}\hat\nabla_{e_\epsilon}(\vartheta^{-1}e_\alpha) = \vartheta^{-2}\hat\nabla_{e_\epsilon}e_\alpha$$

and

$$\widetilde{\mathrm{Rm}}(\bar\nabla_{\tilde e_\epsilon}\tilde e_\alpha,\tilde e_\beta,\tilde e_\gamma,\tilde e_\delta) = \vartheta^{-3}\widehat{\mathrm{Rm}}(\hat\nabla_{e_\epsilon}e_\alpha,e_\beta,e_\gamma,e_\delta).$$

Using (2.9) and (2.10) in (2.8) in any of the cases, we obtain the desired estimate, since $\vartheta'\ge c_{r_0} > 0$ on every interval $[r_0,\infty)$, giving the estimate

$$\vartheta^{-3}\le c\frac{\vartheta'}{\vartheta^3}.$$

□

2.5. *Remark.* For example, if σ is the round metric on $\mathcal{S}_0 = \mathbb{S}^n$, then

$$\tilde R_{\alpha\beta\gamma\delta} = \frac{1}{\vartheta^2}\bar S_{\alpha'\beta'\gamma'\delta'}P^{\alpha'}_\alpha P^{\beta'}_\beta P^{\gamma'}_\gamma P^{\delta'}_\delta.$$

Graphs in warped products. The hypersurfaces

$$x\colon M \hookrightarrow N$$

we deal with in this paper will all be graphs over $\mathcal{S}_0$,

$$x(M) = \{(u(y), y)\colon y \in \mathcal{S}_0\} = \{(u(y(\xi)), y(\xi))\colon \xi \in M\},$$

where

$$u\colon \mathcal{S}_0 \to (R_0, \infty)$$

is smooth. Along M we will always use the *outward* pointing normal

$$\nu = v^{-1}(1, -\vartheta^{-2}\sigma^{ik}u_{;k}),$$

where

$$v^2 = 1 + \vartheta^{-2}\sigma^{ij}u_{;i}u_{;j},$$

and use this normal in the Gaussian formula (2.1). The support function of M is defined by

$$s = \bar{g}(\vartheta\partial_r, \nu) = \frac{\vartheta}{v}. \tag{2.11}$$

There is a relation between the second fundamental form and the graph function on the hypersurface. Let

$$\bar{h} = \vartheta'\vartheta\sigma,$$

then there holds

$$v^{-1}h_{ij} = -u_{;ij} + \bar{h}_{ij},$$

cf. [23, equ. (1.5.10)]. The induced metric is given by

$$g_{ij} = u_{;i}u_{;j} + \vartheta^2\sigma_{ij}$$

and hence

$$v^{-1}h_{ij} = -u_{;ij} + \frac{\vartheta'}{\vartheta}g_{ij} - \frac{\vartheta'}{\vartheta}u_{;i}u_{;j}. \tag{2.12}$$

In order to deduce the gradient estimates, it has proven to be useful to consider the function

$$\varphi\colon \mathcal{S}_0 \to \mathbb{R}$$

$$\varphi(y) = \int_{\inf u_0}^{u(y)} \frac{1}{\vartheta(s)}\, ds. \tag{2.13}$$

There holds

$$h^j_i = \frac{\vartheta'}{\vartheta v}\delta^j_i - \frac{1}{\vartheta v}\tilde{g}^{jk}\varphi_{:ki}, \tag{2.14}$$

where

$$\tilde{g}^{ij} = \sigma^{ij} - \frac{\varphi_{:}{}^{i}\varphi_{:}{}^{j}}{v^2}$$

and the covariant derivative and index raising is performed with respect to σ, cf. [24, equ. (3.26)]. We will use $\hat{\nabla}$ to denote the covariant derivative on $\mathcal{S}_0$ throughout this paper.

2.2. **Curvature functions.** Let $\Gamma \subset \mathbb{R}^n$ be an open and symmetric cone. In Assumption 1.1 the symmetric function $f \in C^\infty(\Gamma)$ is supposed to be evaluated at the principal curvatures of the flow hypersurfaces. This gives rise to an *associated curvature function* F, acting on diagonalisable endomorphisms A of an arbitrary real vector space V via

$$F(A) = f(\mathrm{EV}(A)),$$

where $\mathrm{EV}(A)$ is the unordered n-tuple of eigenvalues of A.

However, when using this definition, F is not defined on the whole space of endomorphisms, but only on the diagonalisable operators. Hence it appears reasonable to view F as defined on bilinear forms,

$$\hat{F}(g,h) := F\left(\frac{1}{2}g^{ik}(h_{kj}+h_{jk})\right)$$

for all positive definite $g = (g_{ij})$ and all bilinear forms $h = (h_{ij}) \in T^{0,2}_p M$. Then

$$\hat{F}^{ij} = \frac{\partial F}{\partial h_{ij}}$$

is a $(2,0)$-tensor and we also write

$$\hat{F}^{ij,kl} = \frac{\partial F}{\partial h_{ij}\partial h_{kl}}.$$

Furthermore, if $F = F(\kappa_i)$ is strictly monotone, then $\hat{F}^{ij}$ is strictly elliptic. If F is concave, then

$$\hat{F}^{ij,kl}\eta_{ij}\eta_{kl} \leq 0$$

for all symmetric (η_{ij}). We refer to [4], [23, Ch. 2] and [61] for more details on curvature functions.

Furthermore we will abuse notation and also write F for $\hat{F}$, since no confusion will be possible. E.g., when writing F^{ij}, we can only mean $\hat{F}^{ij}$, since there are two contravariant indices.

We will also use the special curvature functions H_k, associated to the k-th normalised elementary symmetric polynomial σ_k defined on Γ_k, the connected component of $\{\sigma_k > 0\}$ which contains the point $(1,\dots,1)$.

2.3. **Evolution equations.** The following evolution equations for (1.1) are well known and can be found in several places, for example in [23, Sec. 2.3, Sec. 2.4]. Note that, compared to this reference, we use a different convention on the Riemann tensor.

2.6. **Lemma.** *Denote $\mathcal{F} = -F^{-p}$. Along* (1.1) *there hold:*

(i) The induced metric g satisfies

$$\dot{g} = -2\mathcal{F}h.$$

(ii) The normal vector field satisfies

$$\frac{\bar{\nabla}}{dt}\nu = \operatorname{grad}\mathcal{F},$$

where $\frac{\bar{\nabla}}{dt}$ is the covariant time derivative along the curve $x(\cdot,\xi)$ for fixed $\xi \in M$.

(iii) The second fundamental form satisfies

$$\dot{h}_{ij} = \mathcal{F}_{;ij} - \mathcal{F}h_{ik}h^k_j + \mathcal{F}\overline{\mathrm{Rm}}(x_{;i},\nu,\nu,x_{;j}). \tag{2.15}$$

(iv) The flow speed $\mathcal{F}$ satisfies

$$\dot{\mathcal{F}} - \mathcal{F}^{ij}F_{;ij} = \mathcal{F}^{ij}h_{ik}h^k_j\mathcal{F} + \mathcal{F}^{ij}\overline{\mathrm{Rm}}(x_{;i},\nu,\nu,x_{;j})\mathcal{F}. \tag{2.16}$$

2.7. **Lemma.** *Under the flow* (1.1) *with* $\mathcal{F} = -F^{-p}$ *the second fundamental form evolves by*

$$\begin{aligned}
\dot{h}^i_j - \mathcal{F}^{kl} h^i_{j;kl} &= \mathcal{F}^{kl,rs} h_{kl;}{}^i h_{rs;j} + \mathcal{F}^{kl} h_{rk} h^r_l h^i_j - (\mathcal{F}^{kl} h_{kl} - \mathcal{F}) h^i_r h^r_j \\
&+ \mathcal{F}^{kl} \bar{R}_{\alpha\beta\gamma\delta} \left(x^\alpha_{;l} x^\beta_{;j} x^\gamma_{;k} x^\delta_{;m} h^{im} + x^\alpha_{;l} x^\beta_{;r} x^\gamma_{;k} x^\delta_{;m} h^m_j g^{ri} \right) \\
&+ 2\mathcal{F}^{kl} \bar{R}_{\alpha\beta\gamma\delta} x^\alpha_{;r} x^\beta_{;m} x^\gamma_{;k} x^\delta_{;j} h^m_l g^{ri} - \mathcal{F}^{kl} h_{kl} \bar{R}_{\alpha\beta\gamma\delta} x^\alpha_{;r} \nu^\beta \nu^\gamma x^\delta_{;j} g^{ri} \\
&+ \mathcal{F} \bar{R}_{\alpha\beta\gamma\delta} x^\alpha_{;r} \nu^\beta \nu^\gamma x^\delta_j g^{ri} + \mathcal{F}^{kl} \bar{R}_{\alpha\beta\gamma\delta} x^\alpha_{;k} \nu^\beta \nu^\gamma x^\delta_{;l} h^i_j \\
&+ \mathcal{F}^{kl} \bar{R}_{\alpha\beta\gamma\delta;\epsilon} \nu^\alpha x^\beta_{;k} x^\gamma_{;r} x^\delta_{;l} x^\epsilon_{;j} g^{ri} + \mathcal{F}^{kl} \bar{R}_{\alpha\beta\gamma\delta;\epsilon} \nu^\alpha x^\beta_{;r} x^\gamma_{;j} x^\delta_{;k} x^\epsilon_{;l} g^{ri}.
\end{aligned}$$

Proof. Basically this is [23, Lemma 2.4.1]. For convenience we deduce it again, since the proof in that reference is a little rough and we use another convention for the Riemann tensor. There hold

$$\mathcal{F}_{;i} = \mathcal{F}^{kl} h_{kl;i}$$

and

$$\mathcal{F}_{;ij} = \mathcal{F}^{kl,rs} h_{kl;i} h_{rs;j} + \mathcal{F}^{kl} h_{kl;ij}.$$

We differentiate the Codazzi equation (2.3) to replace the second term on the right hand side. First we differentiate the Codazzi equation with respect to ∂_j, then use the Ricci identities and then differentiate the Codazzi equation with respect to ∂_l. We also use the Weingarten equation (2.2) and the Gauss equation (2.4).

$$\begin{aligned}
h_{kl;ij} &= h_{ki;lj} - \left(\bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_{;k} x^\gamma_{;l} x^\delta_{;i} \right)_{;j} \\
&= h_{ki;jl} + R_{ljk}{}^a h_{ai} + R_{lji}{}^a h_{ka} - \left(\bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_{;k} x^\gamma_{;l} x^\delta_{;i} \right)_{;j} \\
&= R_{ljk}{}^a h_{ai} + R_{lji}{}^a h_{ka} - \left(\bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_{;k} x^\gamma_{;l} x^\delta_{;i} \right)_{;j} \\
&\quad + h_{ij;kl} - \left(\bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_{;i} x^\gamma_{;k} x^\delta_{;j} \right)_{;l} \\
&= h_{ij;kl} + (h_{la} h_{jk} - h_{lk} h_{ja} + \bar{R}_{\alpha\beta\gamma\delta} x^\alpha_{;l} x^\beta_{;j} x^\gamma_{;k} x^\delta_{;a}) h^a_i \\
&\quad + (h_{la} h_{ji} - h_{li} h_{ja} + \bar{R}_{\alpha\beta\gamma\delta} x^\alpha_{;l} x^\beta_{;j} x^\gamma_{;i} x^\delta_{;a}) h^a_k \\
&\quad - \bar{R}_{\alpha\beta\gamma\delta;\epsilon} \nu^\alpha x^\beta_{;k} x^\gamma_{;l} x^\delta_{;i} x^\epsilon_{;j} - \bar{R}_{\alpha\beta\gamma\delta} x^\alpha_{;m} x^\beta_{;k} x^\gamma_{;l} x^\delta_{;i} h^m_j + \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_{;k} \nu^\gamma x^\delta_{;i} h_{lj} \\
&\quad + \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_{;k} x^\gamma_{;l} \nu^\delta h_{ij} - \bar{R}_{\alpha\beta\gamma\delta;\epsilon} \nu^\alpha x^\beta_{;i} x^\gamma_{;k} x^\delta_{;j} x^\epsilon_{;l} - \bar{R}_{\alpha\beta\gamma\delta} x^\alpha_{;m} x^\beta_{;i} x^\gamma_{;k} x^\delta_{;j} h^m_l \\
&\quad + h_{kl} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_{;i} \nu^\gamma x^\delta_{;j} + \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_{;i} x^\gamma_{;k} \nu^\delta h_{jl}.
\end{aligned}$$

Recall that h satisfies (2.15):

$$\dot{h}_{ij} = \mathcal{F}_{;ij} - \mathcal{F} h_{ik} h^k_j + \mathcal{F} \bar{R}_{\alpha\beta\gamma\delta} x^\alpha_i \nu^\beta \nu^\gamma x^\delta_j$$

and hence

$$\begin{aligned}
\dot{h}_{ij} - \mathcal{F}^{kl} h_{ij;kl} &= \mathcal{F}^{kl,rs} h_{kl;i} h_{rs;j} - \mathcal{F}^{kl} h_{kl} h^r_i h_{rj} + \mathcal{F}^{kl} h_{rk} h^r_l h_{ij} \\
&+ \mathcal{F}^{kl} \bar{R}_{\alpha\beta\gamma\delta} \left(x^\alpha_{;l} x^\beta_{;j} x^\gamma_{;k} x^\delta_{;m} h^m_i + x^\alpha_{;l} x^\beta_{;i} x^\gamma_{;k} x^\delta_{;m} h^m_j \right) \\
&+ \mathcal{F}^{kl} h_{kl} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta{}_{;i} \nu^\gamma x^\delta{}_{;j} + \mathcal{F}^{kl} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_{;k} x^\gamma_{;l} \nu^\delta h_{ij} \\
&+ 2\mathcal{F}^{kl} \bar{R}_{\alpha\beta\gamma\delta} x^\alpha_{;i} x^\beta_{;m} x^\gamma_{;k} x^\delta_{;j} h^m_l - \mathcal{F}^{kl} \bar{R}_{\alpha\beta\gamma\delta;\epsilon} \nu^\alpha x^\beta_{;k} x^\gamma_{;l} x^\delta_{;i} x^\epsilon_{;j} \\
&- \mathcal{F}^{kl} \bar{R}_{\alpha\beta\gamma\delta;\epsilon} \nu^\alpha x^\beta_{;i} x^\gamma_{;k} x^\delta_{;j} x^\epsilon_{;l} - \mathcal{F} h_{ik} h^k_j + \mathcal{F} \bar{R}_{\alpha\beta\gamma\delta} x^\alpha_i \nu^\beta \nu^\gamma x^\delta_j.
\end{aligned}$$

The result follows after reverting to the mixed representation. □

Graphical hypersurfaces. Given the flow (1.1) of graphs

$$M_t = \{(u(t,y(t,\xi)), y(t,\xi)) \colon \xi \in M\}$$

in a warped product with metric of the form (2.5), we first of all deduce from (2.12) that

$$\dot{u} - \mathcal{F}^{ij}u_{;ij} = \frac{p+1}{F^p}v^{-1} - \frac{\vartheta'}{\vartheta}\frac{p}{F^{p+1}}F^{ij}g_{ij} + \frac{\vartheta'}{\vartheta}\frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j}. \tag{2.17}$$

Now we deduce the evolution of the quantity

$$w = \frac{1}{\vartheta^2(u)}|du|^2_\sigma = |d\varphi|^2_\sigma,$$

where φ was defined in (2.13). The function φ is better suited to these estimates than u itself, since the representation of the second fundamental form is simpler and so the differentiation of the speed $\mathcal{F}$ is easier to perform. This trick was also used in [22], [65] and in subsequent treatments of graphical expanding flows. Note that φ satisfies

$$\partial_t\varphi = -\mathcal{F}s^{-1}, \tag{2.18}$$

where s is the support function defined in (2.11). In the next lemma we derive the evolution equation for w. We simplify notation: Putting lower indices to a function means covariant differentiation with respect to σ.

2.8. **Lemma.** *Under the flow* (2.18) *in a warped product of the form* (2.5) *the gradient function*

$$|\hat{\nabla}\varphi|^2_\sigma = \varphi_i\varphi^i$$

satisfies

$$\begin{aligned}
&\left(\frac{d}{dt} - \frac{1}{\vartheta^2}\mathcal{F}^k_l\tilde{g}^{lr}\hat{\nabla}_{kr}\right)|\hat{\nabla}\varphi|^2 \\
&= 2\mathcal{F}\frac{s_i}{s^2}\varphi^i - 2\mathcal{F}^k_lh^l_k\frac{s_i}{s^2}\varphi^i + 4\vartheta'\mathcal{F}^k_lh^l_ks^{-1}|\hat{\nabla}\varphi|^2 - 2\frac{\vartheta''}{\vartheta}\mathcal{F}^k_k|\hat{\nabla}\varphi|^2 \\
&-\frac{1}{2v^2\vartheta^2}\mathcal{F}^k_l\sigma^{lm}|\hat{\nabla}\varphi|^2_m|\hat{\nabla}\varphi|^2_k - \frac{1}{v^2\vartheta^2}\mathcal{F}^k_l\varphi^l|\hat{\nabla}\varphi|^2_r\varphi^r_k \\
&+\frac{1}{v^4\vartheta^2}\mathcal{F}^k_l\varphi^l|\hat{\nabla}\varphi|^2_k|\hat{\nabla}\varphi|^2_i\varphi^i - \frac{2}{\vartheta^2}\mathcal{F}^k_l\tilde{g}^{lr}\varphi_{ir}\varphi^i_k - \frac{2}{\vartheta^2}\mathcal{F}^k_l\tilde{g}^{lr}\hat{R}_{ikrm}\varphi^i\varphi^m.
\end{aligned}$$

Proof. From (2.18) we get

$$\frac{d}{dt}|\hat{\nabla}\varphi|^2 = 2\dot{\varphi}_i\varphi^i = \frac{2}{s^2}\mathcal{F}s_i\varphi^i - 2\mathcal{F}^k_l\hat{\nabla}_ih^l_k\varphi^is^{-1}.$$

Due to (2.14) there holds

$$\begin{aligned}
\hat{\nabla}_ih^l_k &= -\frac{v_i\vartheta + v\vartheta'\vartheta\varphi_i}{v^2\vartheta^2}\left(\vartheta'\delta^l_k - \tilde{g}^{lr}\varphi_{rk}\right) + \frac{1}{v\vartheta}\left(\vartheta''\vartheta\varphi_i\delta^l_k - \hat{\nabla}_i\tilde{g}^{lr}\varphi_{rk} - \tilde{g}^{lr}\varphi_{rki}\right) \\
&= -\frac{v_i}{v}h^l_k - \vartheta'\varphi_ih^l_k + \frac{\vartheta''}{v}\delta^l_k\varphi_i + \frac{\varphi^l_i\varphi^r + \varphi^l\varphi^r_i}{v^3\vartheta}\varphi_{rk} - \frac{2}{v^4\vartheta}v_i\varphi^l\varphi^r\varphi_{rk} \\
&\quad - \frac{1}{v\vartheta}\tilde{g}^{lr}\varphi_{rki} \\
&= \frac{s_i}{s}h^l_k - 2\vartheta'\varphi_ih^l_k + \frac{\vartheta''}{v}\delta^l_k\varphi_i + \frac{\varphi^l_i\varphi^r + \varphi^l\varphi^r_i}{v^3\vartheta}\varphi_{rk} \\
&\quad - \frac{1}{v^5\vartheta}|\hat{\nabla}\varphi|^2_i\varphi^l\varphi^r\varphi_{rk} - \frac{1}{v\vartheta}\tilde{g}^{lr}\varphi_{rik} + \frac{1}{v\vartheta}\tilde{g}^{lr}\hat{R}^m_{ikr}\varphi_m,
\end{aligned}$$

where we used the definition of the Riemann tensor of σ. Using

$$|\hat{\nabla}\varphi|^2_{rk} = 2\varphi_{irk}\varphi^i + 2\varphi_{ir}\varphi^i_k,$$

we combine these two equalities to get

$$\begin{aligned}&\left(\frac{d}{dt} - \frac{1}{\vartheta^2}\mathcal{F}^k_l \tilde{g}^{lr}\hat{\nabla}_{kr}\right)|\hat{\nabla}\varphi|^2\\ &= 2\mathcal{F}\frac{s_i}{s^2}\varphi^i - 2\mathcal{F}^k_l h^l_k \frac{s_i}{s^2}\varphi^i + 4\vartheta'\mathcal{F}^k_l h^l_k s^{-1}|\hat{\nabla}\varphi|^2 - 2\frac{\vartheta''}{\vartheta}F^k_k|\hat{\nabla}\varphi|^2\\ &-\frac{2}{v^2\vartheta^2}\mathcal{F}^k_l(\varphi^l_i\varphi^r + \varphi^l\varphi^r_i)\varphi^i\varphi_{rk} + \frac{2}{v^4\vartheta^2}\mathcal{F}^k_l\varphi^l\varphi^r\varphi_{rk}|\hat{\nabla}\varphi|^2_i\varphi^i\\ &-\frac{2}{\vartheta^2}\mathcal{F}^k_l\tilde{g}^{lr}\varphi_{ir}\varphi^i_k - \frac{2}{\vartheta^2}\mathcal{F}^k_l\tilde{g}^{lr}\hat{R}^m_{ikr}\varphi^i\varphi_m\end{aligned}$$

and hence the result. □

The support function satisfies the following evolution.

2.9. **Lemma.** *Along* (1.1) *in a warped product with metric* (2.5)*, the support function*

$$s = \vartheta(u)\bar{g}(\partial_r, \nu)$$

satisfies

$$\dot{s} - \mathcal{F}^{ij}s_{;ij} = \mathcal{F}^{ij}h_{ik}h^k_j s - \vartheta'\frac{p-1}{F^p} + \bar{g}(\vartheta\partial_r, \nabla\mathcal{F}) - \mathcal{F}^{ij}(\bar{g}(\vartheta\partial_r, x_{;k}h^k_{i;j})). \tag{2.19}$$

Proof. The vector field $\vartheta\partial_r$ is conformal,

$$\bar{\nabla}_{\bar{X}}(\vartheta\partial_r) = \vartheta'\bar{X} \quad \forall \bar{X} \in T^{1,0}(N).$$

Hence

$$\dot{s} = \bar{g}(\vartheta'\dot{x}, \nu) + \bar{g}(\vartheta\partial_r, \bar{\nabla}_{\dot{x}}\nu) = -\vartheta'\mathcal{F} + \bar{g}(\vartheta\partial r, \nabla\mathcal{F}),$$

$$Xs = \bar{g}(\vartheta\partial_r, \mathcal{W}(X)) \tag{2.20}$$

and

$$\begin{aligned}\nabla^2 s(X,Y) &= Y(Xs) - (\nabla_Y X)s\\ &= \vartheta' h(X,Y) - h(X, \mathcal{W}(Y))s + \bar{g}(\vartheta\partial_r, \nabla_Y\mathcal{W}(X)).\end{aligned}$$

The result follows from combining these equalities. □

We will also make use of the evolution of $\dot{\varphi}$. This method was used in [5, Prop. 3.4] and [60, Lemma 3.5].

2.10. **Lemma.** *Under the flow* (2.18) *in a warped product of the form* (2.5) *the speed* $\dot{\varphi}$ *satisfies*

$$\partial_t\dot{\varphi} - \frac{\partial\dot{\varphi}}{\partial\varphi_{ij}}\dot{\varphi}_{ij} - \frac{\partial\dot{\varphi}}{\partial\varphi_i}\dot{\varphi}_i = \frac{\vartheta'}{\vartheta}v\mathcal{F}^i_j h^j_i\dot{\varphi} - \frac{\vartheta''}{\vartheta}\mathcal{F}^k_k\dot{\varphi} + \frac{\vartheta'}{\vartheta}v\mathcal{F}\dot{\varphi}. \tag{2.21}$$

Proof. Differentiating

$$\dot{\varphi} = -\mathcal{F}s^{-1}$$

gives

$$\begin{aligned}\partial_t\dot{\varphi} - \frac{\partial\dot{\varphi}}{\partial\varphi_{ij}}\dot{\varphi}_{ij} - \frac{\partial\dot{\varphi}}{\partial\varphi_i}\dot{\varphi}_i &= \frac{\partial\dot{\varphi}}{\partial\varphi}\dot{\varphi}\\ &= -\mathcal{F}^j_i\frac{h^i_j}{\partial\varphi}s^{-1}\dot{\varphi} + s^{-1}\mathcal{F}\vartheta'\dot{\varphi}.\end{aligned}$$

From (2.14) we get

$$\frac{\partial h^i_j}{\partial \varphi} = -\frac{\vartheta'}{v\vartheta}\left(\vartheta'\delta^i_j - \tilde{g}^{ik}\varphi_{kj}\right) + \frac{\vartheta''}{v}\delta^i_j$$

and inserting this gives the result. □

3. Long-time existence

3.1. Barriers.

3.1. **Lemma.** *Let $\vartheta \in C^2((R_0,\infty))$ with $\vartheta' > 0$ and $\vartheta'' \geq 0$, $r_0 > R_0$ and $0 < p \leq 1$. Let $r(t,r_0)$ be the unique solution of the initial value problem*

$$\dot{r} = \frac{\vartheta^p(r)}{n^p\vartheta'^p(r)} \tag{3.1}$$
$$r(0) = r_0.$$

Then r is defined for all times and

$$r(t,r_0) \to \infty, \quad t \to \infty.$$

Consequently, for x_0 as in Theorem 1.3 with associated graph function u_0, we have

$$\inf_M u(t,\cdot) \to \infty, \quad t \to \infty,$$

provided the flow (1.2) *exists for all times.*

Proof. Due to (3.1) we have

$$\dot{r} \leq 1 + \frac{\vartheta(r)}{\vartheta'(r)},$$

where the right hand side grows at most linearly in r due to $\vartheta'' \geq 0$. Hence r is defined for all times. Suppose r does not converge to infinity. Due to its monotonicity it converges to some $r_1 < \infty$. From

$$\frac{\vartheta}{\vartheta'}(r) > 0 \quad \forall r \in [r_0, r_1]$$

we obtain $\dot{r} \geq c > 0$ and reach a contradiction. The second claim follows from the maximum principle which gives

$$r(t, \inf u_0) \leq u(t,\cdot) \leq r(t, \sup u_0).$$

□

3.2. Gradient estimates.

Let us first prove some rough gradient estimates which will suffice to get the long-time existence. In the a priori estimates that appear in the rest of the paper, generic constants will be allowed to depend on the data of the problem, namely N, p, M_0 unless otherwise stated.

First we need a bound on F from below:

3.2. **Lemma.** *Under the assumptions of Theorem 1.3 (i), along* (1.1) *the spatial maxima of the quantity*

$$\dot{\varphi} = \frac{1}{F^p}\frac{v}{\vartheta}$$

are non-increasing.

Proof. According to (2.21) we have

$$\partial_t\dot{\varphi} - \frac{\partial\dot{\varphi}}{\partial\varphi_{ij}}\dot{\varphi}_{ij} - \frac{\partial\dot{\varphi}}{\partial\varphi_i}s^{-1}\dot{\varphi}_i = \frac{(p-1)\vartheta'}{\vartheta F^p}v\dot{\varphi} - \frac{\vartheta'' p}{\vartheta F^{p+1}}F^i_i\dot{\varphi} \leq 0.$$

The result follows from the maximum principle. □

Now we prove some very general gradient estimates for inverse curvature flows in warped products. We use the notation from Lemma 2.8.

3.3. **Lemma.** *Under the assumptions of Theorem 1.3 (i), along* (1.1) *the function* $|\hat{\nabla}\varphi|^2$ *is bounded on every finite time interval. Furthermore, under the assumptions of Theorem 1.3 (ii), there exists a positive constant* γ*, such that the spatial maxima of*

$$\hat{z} = |\hat{\nabla}\varphi|^2\vartheta^\gamma$$

are non-increasing, provided $p < 1$*, and such that the spatial maxima of*

$$\tilde{z} = |\hat{\nabla}\varphi|^2\vartheta'^\gamma$$

are non-increasing, regardless the value of $0 < p \le 1$.

Proof. We want to calculate the evolution equations of $\hat{z}$ and $\tilde{z}$. Hence we need one for u, which makes use of the parabolic operator with respect to the metric σ. Note that in (2.17) we use covariant derivatives of the metric induced by u, hence we need to rewrite this. The covariant derivatives with respect to σ and g are related by

$$\begin{aligned}\hat{\nabla}^2 u &= v^2\nabla^2 u + \frac{\vartheta'}{\vartheta}\left(2du\otimes du - \vartheta^2(v^2-1)\sigma\right)\\ &= -vh + v^2\frac{\vartheta'}{\vartheta}g - v^2\frac{\vartheta'}{\vartheta}du\otimes du + 2\frac{\vartheta'}{\vartheta}du\otimes du - \vartheta'\vartheta(v^2-1)\sigma,\end{aligned}$$

cf. [60, equ. (71)] and (2.12). We obtain

$$\begin{aligned}\left(\partial_t - \frac{1}{\vartheta^2}\frac{p}{F^{p+1}}F^k_l\tilde{g}^{lr}\hat{\nabla}^2_{kr}\right)u &= \frac{v}{F^p} + \frac{pv}{F^p} - \frac{p}{F^{p+1}}\frac{\vartheta'}{\vartheta}F^k_k v^2\\ &\quad + \frac{p}{F^{p+1}}\frac{\vartheta'}{\vartheta}F^{kr}u_{;k}u_{;r}v^2 - \frac{2p}{F^{p+1}}\frac{\vartheta'}{\vartheta}F^{kr}u_{;k}u_{;r}\\ &\quad + \frac{p}{F^{p+1}}\frac{\vartheta'}{\vartheta}F^{kr}(g_{kr} - u_{;k}u_{;r})(v^2-1)\\ &= \frac{p+1}{F^p}v - \frac{p}{F^{p+1}}\frac{\vartheta'}{\vartheta}F^k_k - \frac{p}{F^{p+1}}\frac{\vartheta'}{\vartheta}F^{kr}u_{;k}u_{;r}.\end{aligned}$$

Now we use

$$\frac{s_i}{s} = \vartheta'\varphi_i - \frac{v_i}{v} = \vartheta'\varphi_i - \frac{1}{2}\frac{|\hat{\nabla}\varphi|^2_i}{v^2}$$

to deduce from Lemma 2.8:

$$\begin{aligned}
&\left(\partial_t - \frac{1}{\vartheta^2}\frac{p}{F^{p+1}}F^k_l\tilde g^{lr}\hat\nabla_{kr}\right)|\hat\nabla\varphi|^2\\
={}& -\frac{2}{F^p}\frac{\vartheta'}{\vartheta}v|\hat\nabla\varphi|^2 + \frac{1}{\vartheta v F^p}|\hat\nabla\varphi|^2_i\varphi^i - \frac{2p}{F^p}\frac{\vartheta'}{\vartheta}v|\hat\nabla\varphi|^2 + \frac{p}{\vartheta v F^p}|\hat\nabla\varphi|^2_i\varphi^i\\
&+\frac{4p}{F^p}\frac{\vartheta'}{\vartheta}v|\hat\nabla\varphi|^2 - \frac{2p}{F^{p+1}}\frac{\vartheta''}{\vartheta}F^k_k|\hat\nabla\varphi|^2 - \frac{1}{2v^2\vartheta^2}\frac{p}{F^{p+1}}F^k_l\sigma^{lm}|\hat\nabla\varphi|^2_m|\hat\nabla\varphi|^2_k\\
&-\frac{1}{v^2\vartheta^2}\frac{p}{F^{p+1}}F^k_l\varphi^l|\hat\nabla\varphi|^2_r\varphi^r_k + \frac{1}{v^4\vartheta^2}\frac{p}{F^{p+1}}F^k_l\varphi^l|\hat\nabla\varphi|^2_k|\hat\nabla\varphi|^2_i\varphi^i\\
&-\frac{2}{\vartheta^2}\frac{p}{F^{p+1}}F^k_l\tilde g^{lr}\varphi_{ir}\varphi^i_k - \frac{2}{\vartheta^2}\frac{p}{F^{p+1}}F^k_l\tilde g^{lr}\hat R^m_{ikr}\varphi^i\varphi_m\\
={}& \frac{2(p-1)}{F^p}\frac{\vartheta'}{\vartheta}v|\hat\nabla\varphi|^2 - \frac{2p}{F^{p+1}}\frac{\vartheta''}{\vartheta}F^k_k|\hat\nabla\varphi|^2 - \frac{2}{\vartheta^2}\frac{p}{F^{p+1}}F^k_l\tilde g^{lr}\hat R^m_{ikr}\varphi^i\varphi_m\\
&+\frac{p+1}{F^p}\frac{1}{v\vartheta}|\hat\nabla\varphi|^2_i\varphi^i - \frac{2p}{F^{p+1}}\frac{1}{\vartheta^2}F^k_l\tilde g^{lr}\varphi_{ir}\varphi^i_k - \frac{1}{2v^2\vartheta^2}\frac{p}{F^{p+1}}F^k_l\sigma^{lm}|\hat\nabla\varphi|^2_k|\hat\nabla\varphi|^2_m\\
&+\frac{1}{v^4\vartheta^2}\frac{p}{F^{p+1}}F^k_l\varphi^l|\hat\nabla\varphi|^2_k|\hat\nabla\varphi|^2_i\varphi^i - \frac{1}{v^2\vartheta^2}\frac{p}{F^{p+1}}F^k_l\varphi^l|\hat\nabla\varphi|^2_r\varphi^r_k.
\end{aligned}$$

Now first generally put

$$z = f(u)|\hat\nabla\varphi|^2.$$

With the help of the previous calculations we get at a maximal point of z, where

$$|\hat\nabla\varphi|^2_i = -\frac{f'}{f}\vartheta|\hat\nabla\varphi|^2\varphi_i,$$

$$\begin{aligned}
\mathcal{L}z \equiv{}& \left(\partial_t - \frac{1}{\vartheta^2}\frac{p}{F^{p+1}}F^k_l\tilde g^{lr}\hat\nabla_{kr}\right)z\\
={}& f\mathcal{L}|\hat\nabla\varphi|^2 + z\frac{f'}{f}\mathcal{L}u - \frac{f''}{f}\frac{p}{\vartheta^2F^{p+1}}F^k_l\tilde g^{lr}u_ku_rz - \frac{2p}{\vartheta^2F^{p+1}}F^k_l\tilde g^{lr}|\hat\nabla\varphi|^2_kf_r\\
={}& \frac{2(p-1)}{F^p}\frac{\vartheta'}{\vartheta}vz - \frac{2p}{F^{p+1}}\frac{\vartheta''}{\vartheta}F^k_kz - \frac{2f}{\vartheta^2}\frac{p}{F^{p+1}}F^k_l\tilde g^{lr}\hat R^m_{ikr}\varphi^i\varphi_m\\
&-\frac{p+1}{F^p}\frac{f'}{f}\frac{1}{v}|\hat\nabla\varphi|^2z - \frac{2p}{F^{p+1}}\frac{1}{\vartheta^2}F^k_l\tilde g^{lr}\varphi_{ir}\varphi^i_kf\\
&-\frac{1}{2v^2}\frac{f'^2}{f^2}\frac{p}{F^{p+1}}F^k_l\varphi^l\varphi_k|\hat\nabla\varphi|^2z + \frac{1}{v^4}\frac{f'^2}{f^2}\frac{p}{F^{p+1}}F^k_l\varphi^l\varphi_k|\hat\nabla\varphi|^4z\\
&-\frac{1}{2v^2}\frac{f'^2}{f^2}\frac{p}{F^{p+1}}F^k_l\varphi^l\varphi_k|\hat\nabla\varphi|^2z + \frac{p+1}{F^p}\frac{f'}{f}vz - \frac{p}{F^{p+1}}\frac{f'}{f}\frac{\vartheta'}{\vartheta}F^k_kz\\
&-\frac{p}{F^{p+1}}\frac{f'}{f}\frac{\vartheta'}{\vartheta}F^{kr}u_{;k}u_{;r}z + \frac{2p}{F^{p+1}}\frac{f'^2}{f^2}F^k_l\tilde g^{lr}\varphi_k\varphi_rz - \frac{p}{F^{p+1}}\frac{f''}{f}F^{kr}u_ku_rz\\
={}& \frac{2(p-1)}{F^p}\frac{\vartheta'}{\vartheta}vz - \frac{2p}{F^{p+1}}\frac{\vartheta''}{\vartheta}F^k_kz - \frac{2f}{\vartheta^2}\frac{p}{F^{p+1}}F^k_l\tilde g^{lr}\hat R^m_{ikr}\varphi^i\varphi_m\\
&+\frac{p+1}{F^p}\frac{f'}{f}\frac{1}{v}z - \frac{2p}{F^{p+1}}\frac{1}{\vartheta^2}F^k_l\tilde g^{lr}\varphi_{ir}\varphi^i_kf - \frac{f'^2}{f^2}\frac{p}{F^{p+1}}F^k_l\varphi^l\varphi_k\frac{|\hat\nabla\varphi|^2}{v^4}z\\
&-\frac{p}{F^{p+1}}\frac{f'}{f}\frac{\vartheta'}{\vartheta}F^k_kz + \frac{p}{F^{p+1}}F^{kr}u_ku_rz\left(2\frac{f'^2}{f^2} - \frac{f''}{f} - \frac{f'}{f}\frac{\vartheta'}{\vartheta}\right).
\end{aligned}\tag{3.2}$$

Employing Lemma 3.2 with $f = 1$ we see $\mathcal{L}z \leq cz$, where on each finite interval this constant is bounded. We also used

$$F = n\frac{H_{k+1}}{H_k} \quad \Rightarrow \quad F^k_k \leq c,$$

cf. [49, Lemma 2.7]. Hence we obtain the first claim. Under the assumptions of Theorem 1.3 (ii), we obtain that $\max z$ is decreasing if $p < 1$ and $f(u) = \vartheta^\gamma(u)$ for a small $\gamma > 0$. If $p = 1$, the same is true with $\gamma = 0$. Due to (2.14) we have

$$F = F^k_l h^l_k = \frac{\vartheta'}{v\vartheta}F^k_k - \frac{1}{v\vartheta}F^k_l \tilde{g}^{lr}\varphi_{rk}$$

and hence

$$\begin{aligned}
&\frac{p+1}{F^p}\frac{f'}{f}\frac{1}{v}z - \frac{2p}{F^{p+1}}\frac{1}{\vartheta^2}F^k_l\tilde{g}^{lr}\varphi_{ir}\varphi^i_k f \\
&= \frac{p+1}{F^{p+1}}\frac{f'}{f}\frac{\vartheta'}{\vartheta}\frac{1}{v^2}F^k_k z - \frac{p}{F^{p+1}}\frac{p+1}{p}\frac{f'}{f}\frac{1}{v^2\vartheta}F^k_l\tilde{g}^{lr}\varphi_{rk}z - \frac{2p}{F^{p+1}}\frac{1}{\vartheta^2}F^k_l\tilde{g}^{lr}\varphi_{ir}\varphi^i_k f \\
&= \frac{p+1}{F^{p+1}}\frac{f'}{f}\frac{\vartheta'}{\vartheta}\frac{1}{v^2}F^k_k z + \frac{p}{F^{p+1}}\frac{(p+1)^2}{8p}\frac{f'^2}{f^2}\frac{|\hat{\nabla}\varphi|^2}{v^4}F^k_l\tilde{g}^{lr}\sigma_{rk}z \\
&\quad - \frac{p}{F^{p+1}}\frac{f}{2}F^k_l\tilde{g}^{lr}\left(\frac{p+1}{2p}\frac{f'}{f^2}\frac{1}{v^2}z\sigma_{ir} + \frac{2}{\vartheta}\varphi_{ir}\right)\left(\frac{p+1}{2p}\frac{f'}{f^2}\frac{1}{v^2}z\delta^i_k + \frac{2}{\vartheta}\varphi^i_k\right).
\end{aligned}$$

Hence at a maximal point of z we get for $f = \vartheta'^\gamma$

$$\begin{aligned}
\mathcal{L}z \leq &-\frac{2p}{F^{p+1}}\left(\frac{\vartheta''}{\vartheta} + \frac{1}{2}\frac{f'}{f}\frac{\vartheta'}{\vartheta} - \frac{(p+1)^2}{16p}\frac{f'^2}{f^2}|\hat{\nabla}\varphi|^2 - \frac{p+1}{2p}\frac{f'}{f}\frac{\vartheta'}{\vartheta}\right)F^k_k z \\
&+ \frac{2(p-1)}{F^p}\frac{\vartheta'}{\vartheta}vz - \frac{2f}{\vartheta^2}\frac{p}{F^{p+1}}F^k_l\tilde{g}^{lr}\hat{R}^m_{ikr}\varphi^i\varphi_m \\
&+ \frac{p}{F^{p+1}}F^{kr}u_k u_r z\left(2\frac{f'^2}{f^2} - \frac{f''}{f} - \frac{f'}{f}\frac{\vartheta'}{\vartheta}\right),
\end{aligned} \tag{3.3}$$

which is negative if $0 < \gamma$ is small enough. Here we also used Assumption 1.2. $\square$

3.3. **Curvature estimates.** We prove that along (1.1) all principal curvatures are bounded as long as the flow remains in a compact subset of N. Due to all previous a priori estimates this will imply uniform C^2 estimates on each finite time interval, as well as a uniformly elliptic operator $F^{-(p+1)}F^{ij}$. Hence the regularity estimates by Krylov and Safonov apply to get $C^{2,\alpha}$ estimates. With the linear Schauder estimates we obtain uniform C^∞-bounds on each finite interval. We may extend the solution beyond any finite T, completing the proof of item (i) of Theorem 1.3.

3.4. **Proposition.** *Under the assumptions of Theorem 1.3 (i), on every finite interval* $[0,T]$ *there exists a compact set* $\Lambda \subset \Gamma$ *such that along the flow* (1.1) *the principal curvatures* κ_i *satisfy*

$$(\kappa_i) \in \Lambda \quad \forall 0 \leq t \leq T.$$

Proof. In this proof the generic constant c is allowed to depend on T. We proceed similarly as in [24, 62]. First we simplify the evolution of the second fundamental form, cf. Lemma 2.7. We have the following estimate in normal coordinates:

$$\begin{aligned}
\dot{h}^n_n - \mathcal{F}^{kl}h^n_{n;kl} \leq \mathcal{F}^{kl,rs}h_{kl;n}h_{rs;}{}^n + \frac{p}{F^{p+1}}F^{kl}h_{rk}h^r_l h^n_n - \frac{p+1}{F^p}(h^n_n)^2 \\
+ \frac{c}{F^{p+1}}F^{ij}g_{ij}(h^n_n + 1) + \frac{c}{F^p}.
\end{aligned}$$

According to (2.19) the support function, which is bounded from below due to the gradient estimates Lemma 3.3, satisfies

$$\begin{aligned}\dot{s} - \mathcal{F}^{kl}s_{;kl} = \frac{p}{F^{p+1}}F^{kl}h_{rk}h^r_l s - \frac{(p-1)\vartheta'}{F^p} \\ + \frac{p\vartheta}{F^{p+1}}F^{kl}u_{;}{}^m\overline{\mathrm{Rm}}(\nu, x_{;k}, x_{;m}, x_{;l}).\end{aligned} \tag{3.4}$$

Due to a well known trick, e.g. see the proof of [24, Lemma 4.4], it suffices to bound the evolution equation of the function

$$w = \log h^n_n + f(s) + \alpha u,$$

α to be determined, at a maximal point of w in which normal coordinates are given,

$$g_{ij} = \delta_{ij}, \quad h_{ij} = \kappa_i\delta_{ij}, \quad \kappa_1 \le \cdots \le \kappa_n.$$

For small $\beta > 0$ set

$$f(s) = -\log(s - \beta).$$

Also using (2.17), we see that w satisfies

$$\begin{aligned}\dot{w} - \mathcal{F}^{kl}w_{;kl} \le{}& \frac{p}{F^{p+1}}F^{kl}h_{rk}h^r_l(1 + f's) - \frac{p+1}{F^p}h^n_n \\ &+ \frac{c}{F^{p+1}}F^k_k(1 + (h^n_n)^{-1}) + \frac{c}{F^p}(1 + (h^n_n)^{-1} + \alpha) \\ &+ \frac{p}{F^{p+1}}F^{kl,rs}h_{kl;n}h_{rs;}{}^n(h^n_n)^{-1} + \frac{p}{F^{p+1}}F^{ij}(\log h^n_n)_{;i}(\log h^n_n)_{;j} \\ &- f''\frac{p}{F^{p+1}}F^{ij}s_{;i}s_{;j} - \alpha\frac{\vartheta'}{\vartheta}\frac{p}{F^{p+1}}F^{ij}(g_{ij} - u_{;i}u_{;j}).\end{aligned} \tag{3.5}$$

We employ a trick already used in [19]. Due to the concavity of F there holds

$$F^{nn} \le \cdots \le F^{11} \quad \text{and} \quad F^{kl,rs}\eta_{kl}\eta_{rs} \le \frac{2}{\kappa_n - \kappa_1}\sum_{k=1}^n (F^{nn} - F^{kk})\eta^2_{nk}$$

for all symmetric (η_{ij}). It is possible to exploit this term in order to estimate (3.5).

Case 1: $\kappa_1 < -\epsilon_1\kappa_n$, $0 < \epsilon_1 < \frac{1}{2}$. There hold

$$F^{ij}h_{ik}h^k_j \ge F^{11}\kappa_1^2 \ge \frac{1}{n}F^{ij}g_{ij}\kappa_1^2 \ge \frac{1}{n}F^{ij}g_{ij}\epsilon_1^2\kappa_n^2,$$

$$\begin{aligned}\frac{p}{F^{p+1}}F^{ij}(\log h^n_n)_{;i}(\log h^n_n)_{;j} = f'^2\frac{p}{F^{p+1}}F^{ij}s_{;i}s_{;j} + f'\frac{2\alpha p}{F^{p+1}}F^{ij}s_{;i}u_{;j} \\ + \frac{\alpha^2 p}{F^{p+1}}F^{ij}u_{;i}u_{;j},\end{aligned}$$

due to $\nabla w = 0$. Using (2.20), if κ_n is large, (3.5) with any α becomes

$$\begin{aligned}0 \le{}& \frac{p}{nF^{p+1}}F^{ij}g_{ij}\left(\epsilon_1^2\kappa_n^2(1 + f's) + c + c\alpha|f'|\kappa_n + c\alpha^2\right) \\ &+ \frac{p+1}{F^p}(c + c\alpha - \kappa_n) - \frac{p}{F^{p+1}}F^{ij}s_{;i}s_{;j}(f'' - f'^2) \\ <{}& 0\end{aligned}$$

and hence w is bounded in this case, due to $1 + f's \le c < 0$ and $f'' = f'^2$.

Case 2: $\kappa_1 \geq -\epsilon_1 \kappa_n$. Then

$$
\begin{aligned}
&\frac{2}{\kappa_n - \kappa_1} \sum_{k=1}^{n} (F^{nn} - F^{kk})(h_{nk;n})^2 (h_n^n)^{-1} \\
\leq\ & \frac{2}{1+\epsilon_1} \sum_{k=1}^{n} (F^{nn} - F^{kk})(h_{nk;n})^2 (h_n^n)^{-2} \\
\leq\ & \frac{2}{1+\epsilon_1} \sum_{k=1}^{n} (F^{nn} - F^{kk})(h_{nn;k})^2 (h_n^n)^{-2} + c(\epsilon_1) \sum_{k=1}^{n} (F^{kk} - F^{nn}) \kappa_n^{-2} \\
&\quad + \frac{4}{1+\epsilon_1} \sum_{k=1}^{n} (F^{nn} - F^{kk}) h_{nn;k} \bar{R}_{\alpha\beta\gamma\delta} \nu^a x^\beta_{;n} x^\gamma_{;n} x^\delta_{;k} (h_n^n)^{-2} \\
\leq\ & \frac{2}{1+2\epsilon_1} \sum_{k=1}^{n} (F^{nn} - F^{kk})(h_{nn;k})^2 (h_n^n)^{-2} + c(\epsilon_1) \sum_{k=1}^{n} (F^{kk} - F^{nn}) \kappa_n^{-2},
\end{aligned}
$$

where we used the Codazzi equation (2.3) and Cauchy-Bunjakowski-Schwarz. We deduce further:

$$
\begin{aligned}
&F^{ij} (\log h_n^n)_{;i} (\log h_n^n)_{;j} + \frac{2}{\kappa_n - \kappa_1} \sum_{k=1}^{n} (F^{nn} - F^{kk})(h_{nk;n})^2 (h_n^n)^{-1} \\
\leq\ & \frac{2}{1+2\epsilon_1} \sum_{k=1}^{n} F^{nn} (\log h_n^n)^2_{;k} - \frac{1-2\epsilon_1}{1+2\epsilon_1} \sum_{k=1}^{n} F^{kk} (\log h_n^n)^2_{;k} + c(\epsilon_1) F^{ij} g_{ij} \kappa_n^{-2} \\
\leq\ & \sum_{k=1}^{n} F^{nn} (\log h_n^n)^2_{;k} + c(\epsilon_1) F^{ij} g_{ij} \kappa_n^{-2} \\
=\ & c(\epsilon_1) F^{ij} g_{ij} \kappa_n^{-2} + f'^2 F^{nn} \|\nabla s\|^2 + 2\alpha f' F^{nn} \langle \nabla s, \nabla u \rangle + \alpha^2 F^{nn} \|\nabla u\|^2.
\end{aligned}
$$

Hence we can estimate (3.5), using $F^{ij}\bar{g}_{ij} \geq c_0 F^{ij} g_{ij}$,

$$
\begin{aligned}
0 \leq\ & \frac{p}{F^{p+1}} F^{nn} \left(\kappa_n^2 (1 + f's) + \alpha c \kappa_n + c\alpha^2\right) + \frac{p+1}{F^p} (c + \alpha - \kappa_n) \\
&+ \frac{p}{F^{p+1}} F^{ij} g_{ij} \left(\frac{c(\epsilon_1)}{\kappa_n^2} + c - c_0 \alpha \frac{\vartheta'}{\vartheta} \right) + \frac{p}{F^{p+1}} \left(f'^2 F^{nn} \|\nabla s\|^2 - f'' F^{ij} s_{;i} s_{;j} \right).
\end{aligned}
$$

Due to the barrier estimates, on every finite interval $[0, T]$ there holds $\vartheta \leq c(T)$. Picking α large enough, we see that κ_n is bounded on $[0, T]$ and the proof is complete. $\square$

4. Asymptotics

4.1. **Global bounds.** In order to study the long-time behaviour of (1.1), we need to investigate the evolution of the second fundamental form in greater detail. Therefore we need a more detailed version of its evolution equation.

4.1. **Lemma.** *Along* (1.1) *the Weingarten operator evolves according to*

$$
\begin{aligned}
&\dot{h}^i_j - \frac{p}{F^{p+1}} F^{kl} h^i_{j;kl} \\
=& -\frac{p(p+1)}{F^{p+2}} F_{;j} F_;{}^i + \frac{p}{F^{p+1}} F^{kl,rs} h_{kl;j} h_{rs;}{}^i + \frac{p}{F^{p+1}} F^{kl} h_{rk} h^r_l h^i_j - \frac{p+1}{F^p} h^i_r h^r_j \\
&+ \frac{\vartheta'^2}{\vartheta^2} \frac{p}{F^{p+1}} F^k_k h^i_j - \frac{\vartheta'^2}{\vartheta^2} \frac{p-1}{F^p} \delta^i_j - \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right) v^{-2} \frac{p}{F^{p+1}} F^k_k h^i_j \\
&+ \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right) v^{-2} \frac{p+1}{F^p} \delta^i_j + \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right) \frac{p}{F^{p+1}} F^{kl} \left(u_{;k} u_{;l} h^i_j - 2 h^m_l u_{;m} u_{;k} \delta^i_j\right) \\
&+ \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right) \frac{1-p}{F^p} u_{;j} u_;{}^i + \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right) \frac{p}{F^{p+1}} F^k_k \left(h^i_m u_;{}^m u_{;j} + h^m_j u_{;m} u_;{}^i\right) \\
&+ \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right) \frac{p}{F^{p+1}} \left(F^{il} u_{;m} (h^m_l u_{;j} - h^m_j u_{;l}) + F^k_l h^l_j u_;{}^i u_{;k} - F^l_j h^{im} u_{;l} u_{;m}\right) \\
&+ \frac{p}{F^{p+1}} F^{kl} \left(\widetilde{\mathrm{Rm}}(x_{;l}, x_{;j}, x_{;k}, x_{;m}) h^{im} + \widetilde{\mathrm{Rm}}(x_{;l}, x_{;r}, x_{;k}, x_{;m}) h^m_j g^{ri}\right) \\
&+ \frac{2p}{F^{p+1}} F^{kl} \widetilde{\mathrm{Rm}}(x_{;r}, x_{;m}, x_{;k}, x_{;j}) h^m_l g^{ri} - \frac{p+1}{F^p} \widetilde{\mathrm{Rm}}(x_{;r}, \nu, \nu, x_{;j}) g^{ri} \\
&+ \frac{p}{F^{p+1}} F^{kl} \widetilde{\mathrm{Rm}}(x_{;k}, \nu, \nu, x_{;l}) h^i_j \\
&+ \frac{p}{F^{p+1}} F^{kl} \left(\bar{\nabla}\overline{\mathrm{Rm}}(\nu, x_{;k}, x_{;r}, x_{;l}, x_{;j}) + \bar{\nabla}\overline{\mathrm{Rm}}(\nu, x_{;r}, x_{;j}, x_{;k}, x_{;l})\right) g^{ri}.
\end{aligned}
$$

Proof. In Lemma 2.7 we rewrite the terms involving the Riemann tensor employing (2.6):

$$
\begin{aligned}
&\bar{R}_{\alpha\beta\gamma\delta} x^\alpha_{;l} x^\beta_{;j} x^\gamma_{;k} x^\delta_{;m} \\
=& -\frac{\vartheta''}{\vartheta} (g_{lm} g_{jk} - g_{lk} g_{jm}) + \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right) (\bar{g}_{lm} \bar{g}_{jk} - \bar{g}_{lk} \bar{g}_{jm}) + \tilde{R}_{\alpha\beta\gamma\delta} x^\alpha_{;l} x^\beta_{;j} x^\gamma_{;k} x^\delta_{;m} \\
=& -\frac{\vartheta'^2}{\vartheta^2} (g_{lm} g_{jk} - g_{lk} g_{jm}) + \tilde{R}_{\alpha\beta\gamma\delta} x^\alpha_{;l} x^\beta_{;j} x^\gamma_{;k} x^\delta_{;m} \\
&- \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right) (u_{;l} u_{;m} g_{jk} + u_{;j} u_{;k} g_{lm} - u_{;l} u_{;k} g_{jm} - u_{;j} u_{;m} g_{lk}).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\mathcal{F}^{kl} \bar{R}_{\alpha\beta\gamma\delta} \left(x^\alpha_{;l} x^\beta_{;j} x^\gamma_{;k} x^\delta_{;m} h^{im} + x^\alpha_{;l} x^\beta_{;r} x^\gamma_{;k} x^\delta_{;m} h^m_j g^{ri}\right) \\
=& -\frac{\vartheta'^2}{\vartheta^2} \frac{p}{F^{p+1}} F^{kl} h^i_l g_{jk} + \frac{\vartheta'^2}{\vartheta^2} \frac{p}{F^{p+1}} F^k_k h^i_j - \frac{\vartheta'^2}{\vartheta^2} \frac{p}{F^{p+1}} F^{il} h_{jl} + \frac{\vartheta'^2}{\vartheta^2} \frac{p}{F^{p+1}} F^k_k h^i_j \\
&+ \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right) \frac{p}{F^{p+1}} F^{kl} (u_{;k} u_{;l} h^i_j + g_{kl} h^{im} u_{;m} u_{;j} - h^i_l u_{;j} u_{;k} - g_{kj} h^{im} u_{;l} u_{;m}) \\
&+ \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right) \frac{p}{F^{p+1}} (F^{kl} (u_{;k} u_{;l} h^i_j + g_{kl} h^m_j u_{;m} u_;{}^i) - F^{il} h^m_j u_{;l} u_{;m} - F^k_m h^m_j u_;{}^i u_{;k}) \\
&+ \frac{p}{F^{p+1}} F^{kl} \left(\tilde{R}_{\alpha\beta\gamma\delta} x^\alpha_{;l} x^\beta_{;j} x^\gamma_{;k} x^\delta_{;m} h^{im} + \tilde{R}_{\alpha\beta\gamma\delta} x^\alpha_{;l} x^\beta_{;r} x^\gamma_{;k} x^\delta_{;m} h^m_j g^{ri}\right)
\end{aligned}
$$

and thus the following two equations hold:

$$
\begin{aligned}
&\mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta}\left(x^\alpha_{;l}x^\beta_{;j}x^\gamma_{;k}x^\delta_{;m}h^{im} + x^\alpha_{;l}x^\beta_{;r}x^\gamma_{;k}x^\delta_{;m}h^m_j g^{ri}\right)\\
=&\ -2\frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}(F^i_l h^l_j - F^{kl}g_{kl}h^i_j)\\
&+\left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}(2F^{kl}u_{;k}u_{;l}h^i_j + F^{kl}g_{kl}(h^{im}u_{;m}u_{;j} + h^m_j u_{;m}u_;{}^i)\\
&\qquad - F^l_j h^{im}u_{;l}u_{;m} - F^{il}h^m_j u_{;l}u_{;m} - F^{kl}h^i_l u_{;j}u_{;k} - F^k_m h^m_j u_;{}^i u_{;k})\\
&+\frac{p}{F^{p+1}}F^{kl}\left(\tilde{R}_{\alpha\beta\gamma\delta}x^\alpha_{;l}x^\beta_{;j}x^\gamma_{;k}x^\delta_{;m}h^{im} + \tilde{R}_{\alpha\beta\gamma\delta}x^\alpha_{;l}x^\beta_{;r}x^\gamma_{;k}x^\delta_{;m}h^m_j g^{ri}\right),
\end{aligned}
$$

$$
\begin{aligned}
&2\mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;r}x^\beta_{;m}x^\gamma_{;k}x^\delta_{;j}h^m_l g^{ri}\\
=&\ -2\frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}(F\delta^i_j - F^i_l h^l_j) + \frac{2p}{F^{p+1}}F^{kl}\tilde{R}_{\alpha\beta\gamma\delta}x^\alpha_{;r}x^\beta_{;m}x^\gamma_{;k}x^\delta_{;j}h^m_l g^{ri}\\
&+2\left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}(F^k_l h^l_j u_;{}^i u_{;k} + F^{il}h^m_l u_{;j}u_{;m} - Fu_;{}^i u_{;j} - F^{kl}h^m_l u_{;m}u_{;k}\delta^i_j).
\end{aligned}
$$

Adding up, also using $F^i_k h^k_j = h^i_k F^k_j$, gives

$$
\begin{aligned}
&\mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta}\left(x^\alpha_{;l}x^\beta_{;j}x^\gamma_{;k}x^\delta_{;m}h^{im} + x^\alpha_{;l}x^\beta_{;r}x^\gamma_{;k}x^\delta_{;m}h^m_j g^{ri}\right)\\
&+2\mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;r}x^\beta_{;m}x^\gamma_{;k}x^\delta_{;j}h^m_l g^{ri}\\
&=2\frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}(F^k_k h^i_j - F\delta^i_j)\\
&+\left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}(2F^{kl}u_{;k}u_{;l}h^i_j + F^k_k(h^i_m u_;{}^m u_{;j} + h^m_j u_{;m}u_;{}^i)\\
&\qquad - F^l_j h^{im}u_{;l}u_{;m} - F^{il}h^m_j u_{;l}u_{;m} + F^k_l h^l_j u_;{}^i u_{;k}\\
&\qquad + F^{il}h^m_l u_{;j}u_{;m} - 2Fu_;{}^i u_{;j} - 2F^{kl}h^m_l u_{;m}u_{;k}\delta^i_j)\\
&+\frac{p}{F^{p+1}}F^{kl}\tilde{R}_{\alpha\beta\gamma\delta}\left(x^\alpha_{;l}x^\beta_{;j}x^\gamma_{;k}x^\delta_{;m}h^{im} + x^\alpha_{;l}x^\beta_{;r}x^\gamma_{;k}x^\delta_{;m}h^m_j g^{ri}\right)\\
&+\frac{2p}{F^{p+1}}F^{kl}\tilde{R}_{\alpha\beta\gamma\delta}x^\alpha_{;r}x^\beta_{;m}x^\gamma_{;k}x^\delta_{;j}h^m_l g^{ri}.
\end{aligned}
\tag{4.1}
$$

Using $\nu = v^{-1}(1, -\bar{g}^{ik}u_{;k})$ and $v^{-2}\bar{g}^{ij}u_{;i}u_{;j} = \|\nabla u\|^2$, we get

$$
\begin{aligned}
\overline{\mathrm{Rm}}(x_{;i},\nu,\nu,x_{;j}) &= -\frac{\vartheta''}{\vartheta}g_{ij} + \left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)(\|\nabla u\|^2\bar{g}_{ij} - v^{-2}u_{;i}u_{;j})\\
&\quad + \widetilde{\mathrm{Rm}}(x_{;i},\nu,\nu,x_{;j})\\
&= -\frac{\vartheta''}{\vartheta}g_{ij} + \left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^2}{\vartheta^2}\right)(\|\nabla u\|^2 g_{ij} - u_{;i}u_{;j})\\
&\quad + \widetilde{\mathrm{Rm}}(x_{;i},\nu,\nu,x_{;j}).
\end{aligned}
\tag{4.2}
$$

Thus

$$\begin{aligned}
&(\mathcal{F}-\mathcal{F}^{kl}h_{kl})\bar{R}_{\alpha\beta\gamma\delta}x^{\alpha}_{;r}\nu^{\beta}\nu^{\gamma}x^{\delta}_{j}g^{ri}+\mathcal{F}^{kl}\bar{R}_{\alpha\beta\gamma\delta}x^{\alpha}_{;k}\nu^{\beta}\nu^{\gamma}x^{\delta}_{;l}h^{i}_{j}\\
&=\frac{p+1}{F^{p}}\left(\frac{\vartheta''}{\vartheta}\delta^{i}_{j}-\left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^{2}}{\vartheta^{2}}\right)(\|\nabla u\|^{2}\delta^{i}_{j}-u_{;j}u_{;}{}^{i})\right)\\
&\quad-\frac{p}{F^{p+1}}F^{kl}\left(\frac{\vartheta''}{\vartheta}g_{kl}-\left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^{2}}{\vartheta^{2}}\right)(\|\nabla u\|^{2}g_{kl}-u_{;l}u_{;k})\right)h^{i}_{j}\\
&\quad-\frac{p+1}{F^{p}}\widetilde{\mathrm{Rm}}(x_{;r},\nu,\nu,x_{;j})g^{ri}+\frac{p}{F^{p+1}}F^{kl}\widetilde{\mathrm{Rm}}(x_{;k},\nu,\nu,x_{;l})h^{i}_{j}.
\end{aligned}\tag{4.3}$$

Adding up (4.1) and (4.3) and inserting the result into Lemma 2.7 gives the claimed formula. ☐

4.2. **Lemma.** *Along* (1.1) *the function* $v=\vartheta s^{-1}$ *satisfies the evolution equation*

$$\begin{aligned}
&\dot{v}-\frac{p}{F^{p+1}}F^{ij}v_{;ij}\\
&=-\frac{p}{F^{p+1}}F^{ij}h_{ik}h^{k}_{j}v-\frac{\vartheta'^{2}}{\vartheta^{2}}\frac{p}{F^{p+1}}F^{ij}g_{ij}v+\frac{\vartheta'}{\vartheta}\frac{p+1}{F^{p}}+\frac{\vartheta'}{\vartheta}\frac{p-1}{F^{p}}v^{2}\\
&\quad+\left(\frac{\vartheta'^{2}}{\vartheta^{2}}-\frac{\vartheta''}{\vartheta}\right)\frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j}v\\
&\quad+\frac{p}{F^{p+1}}F^{kl}\left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^{2}}{\vartheta^{2}}\right)(u_{;l}u_{;k}-\|\nabla u\|^{2}g_{kl})v\\
&\quad-\frac{p}{F^{p+1}}F^{kl}\widetilde{\mathrm{Rm}}(x_{;k},\nu,\nu,x_{;l})v+2\frac{\vartheta'}{\vartheta}\frac{p}{F^{p+1}}F^{ij}u_{;i}v_{;j}-\frac{2}{v}\frac{p}{F^{p+1}}F^{ij}v_{;i}v_{;j}.
\end{aligned}\tag{4.4}$$

Proof. Due to (2.17) and (3.4) we have

$$\begin{aligned}
&\dot{v}-\frac{p}{F^{p+1}}F^{ij}v_{;ij}\\
&=\frac{\vartheta'}{\vartheta}v\left(\dot{u}-\frac{p}{F^{p+1}}F^{ij}u_{;ij}\right)-\frac{\vartheta''}{\vartheta}v\frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j}-\frac{v}{s}\left(\dot{s}-\frac{p}{F^{p+1}}F^{ij}s_{;ij}\right)\\
&\quad-\frac{2\vartheta}{s^{3}}\frac{p}{F^{p+1}}F^{ij}s_{;i}s_{;j}-2\frac{p}{F^{p+1}}F^{ij}\vartheta_{;i}\left(\frac{1}{s}\right)_{;j}\\
&=\frac{\vartheta'}{\vartheta}\frac{p+1}{F^{p}}-\frac{\vartheta'^{2}}{\vartheta^{2}}\frac{p}{F^{p+1}}F^{ij}g_{ij}v+\left(\frac{\vartheta'^{2}}{\vartheta^{2}}-\frac{\vartheta''}{\vartheta}\right)\frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j}v\\
&\quad-\frac{p}{F^{p+1}}F^{ij}h_{ik}h^{k}_{j}v+\frac{\vartheta'}{\vartheta}\frac{p-1}{F^{p}}v^{2}-\frac{p}{F^{p+1}}F^{kl}\overline{\mathrm{Rm}}(\nu,x_{;k},x_{;m},x_{;l})u_{;}{}^{m}v^{2}\\
&\quad-\frac{2}{v}\frac{p}{F^{p+1}}F^{ij}v_{;i}v_{;j}-2v\frac{\vartheta'^{2}}{\vartheta^{2}}\frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j}+4\frac{\vartheta'}{\vartheta}\frac{p}{F^{p+1}}F^{ij}u_{;i}v_{;j}\\
&\quad-2\frac{\vartheta'}{\vartheta}\frac{p}{F^{p+1}}F^{ij}u_{;i}v_{;j}+2\frac{\vartheta'^{2}}{\vartheta^{2}}\frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j}v,
\end{aligned}$$

which is the claimed formula up to rewriting the term involving $\overline{\mathrm{Rm}}$. However, we use (2.6) to deduce

$$\begin{aligned}
\overline{\mathrm{Rm}}(\nu,x_{;k},x_{;m},x_{;l})u_{;}{}^{m}&=-\left(\frac{\vartheta''}{\vartheta}-\frac{\vartheta'^{2}}{\vartheta^{2}}\right)v^{-1}(u_{;l}u_{;k}-\|\nabla u\|^{2}g_{kl})\\
&\quad+\widetilde{\mathrm{Rm}}(\nu,x_{;k},x_{;m},x_{;l})u_{;}{}^{m}
\end{aligned}$$

and hence

$$\begin{aligned}\overline{\mathrm{Rm}}(\nu, x_{;k}, x_{;m}, x_{;l}) u_;{}^m v^2 = &-\left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)(u_{;l}u_{;k} - \|\nabla u\|^2 g_{kl})v \\ &+ \widetilde{\mathrm{Rm}}(x_{;k}, \nu, \nu, x_{;l})v,\end{aligned}$$

where we have used

$$u_;{}^m = v^{-2}\vartheta^{-2}\sigma^{mk}u_{;k}.$$

Inserting gives the result. □

We start the investigation of the long-time behaviour of (1.2) under the assumptions in item (ii) of Theorem 1.3 by proving a lower bound on the curvature function.

4.3. **Lemma.** *Under the assumptions of Theorem 1.3 (ii), along* (1.1) *there exists a constant* c*, such that*

$$\frac{\vartheta' v}{\vartheta F} \le c.$$

Proof. If $p = 1$ and ϑ' is bounded , the result follows from Lemma 3.2 immediately. If $p < 1$ or ϑ' is unbounded, Lemma 3.3 says that $v \to 1$. Due (2.16), (2.17), (4.2) and (4.4) the function

$$w = \log\left(\frac{1}{F^p}\right) + f(v) + p\log\vartheta' - p\log\vartheta,$$

where f with $f' \ge 0$ is yet to be determined, satisfies

$$\begin{aligned}&\dot w - \frac{p}{F^{p+1}}F^{ij}w_{;ij} \\ &= \frac{p}{F^{p+1}}F^{ij}h_{ik}h^k_j - \frac{\vartheta''}{\vartheta}\frac{p}{F^{p+1}}F^{ij}g_{ij} + \frac{p}{F^{p+1}}F^{ij}\widetilde{\mathrm{Rm}}(x_{;i}, \nu, \nu, x_{;j}) \\ &+ \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}F^{ij}(\|\nabla u\|^2 g_{ij} - u_{;i}u_{;j}) \\ &+ \frac{p}{F^{p+1}}F^{ij}\left(\log\frac{1}{F^p}\right)_{;i}\left(\log\frac{1}{F^p}\right)_{;j} - \frac{p}{F^{p+1}}F^{ij}h_{ik}h^k_j f'v - \frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}F^{ij}g_{ij}f'v \\ &+ \frac{\vartheta'}{\vartheta}\frac{p+1}{F^p}f' + \frac{\vartheta'}{\vartheta}\frac{p-1}{F^p}f'v^2 + \left(\frac{\vartheta'^2}{\vartheta^2} - \frac{\vartheta''}{\vartheta}\right)\frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j}f'v \\ &+ \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}F^{ij}(u_{;i}u_{;j} - \|\nabla u\|^2 g_{ij})f'v \\ &- \frac{p}{F^{p+1}}F^{ij}\widetilde{\mathrm{Rm}}(x_{;i}, \nu, \nu, x_{;j})f'v + 2f'\frac{\vartheta'}{\vartheta}\frac{p}{F^{p+1}}F^{ij}u_{;i}v_{;j} - \frac{2}{v}f'\frac{p}{F^{p+1}}F^{ij}v_{;i}v_{;j} \\ &- f''\frac{p}{F^{p+1}}F^{ij}v_{;i}v_{;j} + (p+1)\left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta}\right)\frac{p}{F^{p+1}}v^{-1}F \\ &- p\left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}F^{ij}g_{ij} + p\left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j} \\ &- p\left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta}\right)'\frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j}.\end{aligned}$$

Sorting the terms appropriately and replacing $(\log 1/F^p)_{;i}$ we get

$$
\begin{aligned}
&\dot{w} - \frac{p}{F^{p+1}} F^{ij} w_{;ij} \\
&\leq \frac{p}{F^{p+1}} F^{ij} h_{ik} h_j^k (1 - f'v) + \frac{p}{F^{p+1}} F^{ij} \widetilde{\mathrm{Rm}}(x_{;i}, \nu, \nu, x_{;j})(1 - f'v) \\
&+ \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right) \frac{p}{F^{p+1}} F^{ij} (\|\nabla u\|^2 g_{ij} - u_{;i} u_{;j})(1 - f'v) \\
&+ \frac{p(p+1)}{F^{p+1}} \frac{\vartheta''}{\vartheta} \left(-F^{ij} g_{ij} + \frac{\vartheta}{\vartheta'} F v^{-1} + c(f'v + 1) F^{ij} u_{;i} u_{;j}\right) \\
&+ \frac{\vartheta'^2}{\vartheta^2} \frac{p}{F^{p+1}} \Big(- F^{ij} g_{ij} f'v + \frac{\vartheta}{\vartheta'} \frac{p+1}{p} f'F + \frac{\vartheta}{\vartheta'} \frac{p-1}{p} f'Fv^2 \\
&- (p+1) \frac{\vartheta}{\vartheta'} F v^{-1} + pF^{ij} g_{ij} \Big) + \frac{\vartheta'^2}{\vartheta^2} \frac{p}{F^{p+1}} F^{ij} u_{;i} u_{;j} (f'v - 1) \\
&+ (p-1)^2 \frac{\vartheta'^2}{\vartheta^2} \frac{p}{F^{p+1}} F^{ij} u_{;i} u_{;j} + \frac{p}{F^{p+1}} F^{ij} w_{;i} w_{;j} - 2f' \frac{p}{F^{p+1}} F^{ij} w_{;i} v_{;j} \\
&- 2p \left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta}\right) \frac{p}{F^{p+1}} F^{ij} w_{;i} u_{;j} + f'^2 \frac{p}{F^{p+1}} F^{ij} v_{;i} v_{;j} \\
&- 2pf' \left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta}\right) \frac{p}{F^{p+1}} F^{ij} v_{;i} u_{;j} + 2f' \frac{\vartheta'}{\vartheta} \frac{p}{F^{p+1}} F^{ij} u_{;i} v_{;j} \\
&- \frac{2}{v} f' \frac{p}{F^{p+1}} F^{ij} v_{;i} v_{;j} - f'' \frac{p}{F^{p+1}} F^{ij} v_{;i} v_{;j}.
\end{aligned}
\tag{4.5}
$$

Now choose

$$
f(v) = -\log\left(v^{-\frac{1}{2}} - \frac{3}{4}\right)
$$

where in the sequel we only consider sufficiently large times where $v^{\frac{1}{2}} < \frac{4}{3}$. Then

$$
f' = \frac{1}{2} \frac{v^{-\frac{3}{2}}}{v^{-\frac{1}{2}} - \frac{3}{4}}, \quad f'' = -\frac{3}{4} \frac{v^{-\frac{5}{2}}}{v^{-\frac{1}{2}} - \frac{3}{4}} + \frac{1}{4} \frac{v^{-3}}{\left(v^{-\frac{1}{2}} - \frac{3}{4}\right)^2}, \quad f'v = \frac{1}{2} \frac{v^{-\frac{1}{2}}}{v^{-\frac{1}{2}} - \frac{3}{4}} \geq \frac{3}{2}
$$

and

$$
f'^2 - \frac{2}{v} f' - f'' = -\frac{v^{-\frac{5}{2}}}{v^{-\frac{1}{2}} - \frac{3}{4}} + \frac{3}{4} \frac{v^{-\frac{5}{2}}}{v^{-\frac{1}{2}} - \frac{3}{4}} = -\frac{1}{4} \frac{v^{-\frac{5}{2}}}{v^{-\frac{1}{2}} - \frac{3}{4}} \leq -\frac{3}{4v^2}.
$$

Hence, using Cauchy-Schwarz on $F^{ij} u_{;i} v_{;j}$, we can estimate further:

$$
\begin{aligned}
&\dot{w} - \frac{p}{F^{p+1}} F^{ij} w_{;ij} \\
&\leq \frac{p(p+1)}{F^{p+1}} \frac{\vartheta''}{\vartheta} \left(-F^{ij} g_{ij} + \frac{\vartheta}{\vartheta'} F v^{-1} + c_\epsilon (f'v + 1) F^{ij} u_{;i} u_{;j}\right) \\
&+ \frac{\vartheta'^2}{\vartheta^2} \frac{p}{F^{p+1}} \Big(pF^{ij} g_{ij} - F^{ij} g_{ij} f'v + 2\frac{\vartheta}{\vartheta'} f'F - (p+1) \frac{\vartheta}{\vartheta'} F v^{-1} + F^{ij} g_{ij} \|\nabla u\|^2 f'v \\
&+ c_\epsilon (f'v + 1) F^{ij} u_{;i} u_{;j} \Big) + \frac{p}{F^{p+1}} F^{ij} v_{;i} v_{;j} \left(-\frac{3}{4v^2} + \frac{\epsilon c}{v} f'\right) \\
&+ \frac{p}{F^{p+1}} F^{ij} w_{;i} w_{;j} - 2f' \frac{p}{F^{p+1}} F^{ij} w_{;i} v_{;j} - 2p \left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta}\right) \frac{p}{F^{p+1}} F^{ij} w_{;i} u_{;j} \\
&< 0
\end{aligned}
$$

at maximal points of w, if ϵ is chosen small, $\|\nabla u\|$ is small enough (which happens eventually) and w is large. Hence w is bounded. □

We need a similar estimate of the rescaled principal curvatures.

4.4. **Lemma.** *Under the assumptions of Theorem 1.3 (ii), along* (1.1) *there exists a constant c, such that*

$$\kappa_n \le c\frac{\vartheta'}{\vartheta}.$$

Proof. Define

$$z = \log h^n_n + \log\frac{\vartheta}{\vartheta'} + f(v).$$

We estimate the evolution of z directly from (2.17), Lemma 4.1 and (4.4) and, as in the proof of Proposition 3.4, from the start calculate in a maximal point of z in coordinates such that

$$g_{ij} = \delta_{ij}, \quad h_{ij} = \kappa_i\delta_{ij}, \quad \kappa_1 \le \cdots \le \kappa_n.$$

First of all there holds

$$\begin{aligned} &F^{kl}\widetilde{\mathrm{Rm}}(x_{;l}, x_{;j}, x_{;k}, x_{;m})h^{im} + F^{kl}\widetilde{\mathrm{Rm}}(x_{;l}, x_{;r}, x_{;k}, x_{;m})h^m_j g^{ri} \\ &+ 2F^{kl}\widetilde{\mathrm{Rm}}(x_{;r}, x_{;m}, x_{;k}, x_{;j})h^m_l g^{ri} \\ &= 2F^{kk}\widetilde{\mathrm{Rm}}(x_{;n}, x_{;k}, x_{;k}, x_{;n})(\kappa_k - \kappa_n) \le 0, \end{aligned}$$

since the sectional curvatures of σ are non-negative.

Due to Lemma 2.4 we have

$$\|\widetilde{\mathrm{Rm}}\| \le \frac{c}{\vartheta^2}, \quad \|\bar\nabla\widetilde{\mathrm{Rm}}\| \le c\frac{\vartheta'}{\vartheta^3},$$

and we get

$$\begin{aligned} &F^{kl}\bar\nabla\overline{\mathrm{Rm}}(\nu, x_{;k}, x_{;r}, x_{;l}, x_{;j})g^{ri} + F^{kl}\bar\nabla\overline{\mathrm{Rm}}(\nu, x_{;r}, x_{;j}, x_{;k}, x_{;l})g^{ri} \\ &\le c\frac{\vartheta'^3}{\vartheta^3}\|\nabla u\|^2 F^k_k + c\frac{\vartheta'}{\vartheta^3}\|\nabla u\| F^k_k, \end{aligned}$$

where we have used that the terms in (2.7) involving $r_{;\alpha}$ are cancelled , since $\bar T$ carries the symmetries of a curvature tensor.

Hence

$$\begin{aligned} &\dot z - \frac{p}{F^{p+1}}F^{ij}z_{;ij} \\ &\le \frac{p}{F^{p+1}}F^{ij}(\log h^n_n)_{;i}(\log h^n_n)_{;j} + \frac{p}{F^{p+1}}F^{ij}h_{ik}h^k_j(1-f'v) - \frac{p+1}{F^p}h^n_n \\ &+ \frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}F^k_k(1-f'v) - \frac{\vartheta'^2}{\vartheta^2}\frac{p-1}{F^p}\kappa_n^{-1} - \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)\frac{p}{v^2F^{p+1}}F^k_k \\ &+ \left(\frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2}\right)\frac{p+1}{v^2F^p}\kappa_n^{-1} + c(1+|f'|)\frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}F^k_k\|\nabla u\|^2 \\ &+ c\frac{\vartheta'^2}{\vartheta^2}\frac{1}{F^p}\|\nabla u\|^2\kappa_n^{-1} + \frac{c}{F^{p+1}}\frac{\vartheta'^3}{\vartheta^3}\|\nabla u\|^2\kappa_n^{-1}F^k_k + \frac{c}{F^{p+1}}\frac{\vartheta'}{\vartheta^3}\|\nabla u\|\kappa_n^{-1}F^k_k \\ &+ f'\frac{\vartheta'}{\vartheta}\frac{p+1}{F^p} + f'\frac{\vartheta'}{\vartheta}\frac{p-1}{F^p}v^2 + 2f'\frac{\vartheta'}{\vartheta}\frac{p}{F^{p+1}}F^{ij}u_{;i}v_{;j} - \frac{2f'}{v}\frac{p}{F^{p+1}}F^{ij}v_{;i}v_{;j} \\ &- f''\frac{p}{F^{p+1}}F^{ij}v_{;i}v_{;j} + \left(\frac{\vartheta'}{\vartheta} - \frac{\vartheta''}{\vartheta'}\right)\frac{p+1}{F^p}v^{-1} - \left(\frac{\vartheta'^2}{\vartheta^2} - \frac{\vartheta''}{\vartheta}\right)\frac{p}{F^{p+1}}F^k_k. \end{aligned}$$

Hence

$$
\begin{aligned}
&\dot z - \frac{p}{F^{p+1}} F^{ij} z_{;ij} \\
\le\, & \frac{p}{F^{p+1}} F^{ij} (\log h^n_n)_{;i} (\log h^n_n)_{;j} + \frac{p}{F^{p+1}} F^{ij} h_{ik} h^k_j (1 - f'v) \\
&+ \frac{\vartheta'^2}{\vartheta^2} \frac{p}{F^{p+1}} F^k_k (1 - f'v) - \frac{p+1}{F^p} \left(\kappa_n - \frac{\vartheta'}{\vartheta} f' \right) \\
&+ \left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta} \right) \frac{p+1}{F^p} v^{-1} \left(v^{-1} \kappa_n^{-1} \frac{\vartheta'}{\vartheta} - 1 \right) + \frac{\vartheta'}{\vartheta} \frac{p-1}{F^p} \left(f' v^2 - \frac{\vartheta'}{\vartheta} \kappa_n^{-1} \right) \\
&+ \frac{\vartheta'^2}{\vartheta^2} \left(\frac{\vartheta'}{\vartheta} \kappa_n^{-1} + 1 \right) \frac{c(1 + |f'|)}{F^{p+1}} F^k_k \|\nabla u\|^2 + \frac{\vartheta'^2}{\vartheta^2} \frac{c}{F^p} \|\nabla u\|^2 \kappa_n^{-1} \\
&+ \frac{c}{F^{p+1}} \frac{\vartheta'}{\vartheta^3} \|\nabla u\| \kappa_n^{-1} F^k_k + 2 f' \frac{\vartheta'}{\vartheta} \frac{p}{F^{p+1}} F^{ij} u_{;i} v_{;j} - \frac{2f'}{v} \frac{p}{F^{p+1}} F^{ij} v_{;i} v_{;j} \\
&- f'' \frac{p}{F^{p+1}} F^{ij} v_{;i} v_{;j}.
\end{aligned} \tag{4.6}
$$

Furthermore we insert

$$
(\log h^n_n)_{;i} = z_{;i} - \left(\frac{\vartheta'}{\vartheta} - \frac{\vartheta''}{\vartheta'} \right) u_{;i} - f' v_{;i}
$$

and hence

$$
\begin{aligned}
&\dot z - \frac{p}{F^{p+1}} F^{ij} z_{;ij} \\
\le\, & \frac{p}{F^{p+1}} F^{ij} h_{ik} h^k_j (1 - f'v) + \frac{\vartheta'^2}{\vartheta^2} \frac{p}{F^{p+1}} F^k_k (1 - f'v) - \frac{p+1}{F^p} \left(\kappa_n - \frac{\vartheta'}{\vartheta} f' \right) \\
&+ \left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta} \right) \frac{p+1}{F^p} v^{-1} \left(v^{-1} \kappa_n^{-1} \frac{\vartheta'}{\vartheta} - 1 \right) + \frac{p-1}{F^p} \left(\frac{\vartheta'}{\vartheta} f' v^2 - \frac{\vartheta'^2}{\vartheta^2} \kappa_n^{-1} \right) \\
&+ \frac{\vartheta'^2}{\vartheta^2} \left(\frac{\vartheta'}{\vartheta} \kappa_n^{-1} + 1 \right) \frac{c(1 + |f'|)}{F^{p+1}} F^k_k \|\nabla u\|^2 + \frac{\vartheta'^2}{\vartheta^2} \frac{c}{F^p} \|\nabla u\|^2 \kappa_n^{-1} \\
&+ \frac{c}{F^{p+1}} \frac{\vartheta'}{\vartheta^3} \|\nabla u\| \kappa_n^{-1} F^k_k + 2 f' \frac{\vartheta'}{\vartheta} \frac{p}{F^{p+1}} F^{ij} u_{;i} v_{;j} \\
&+ \frac{p}{F^{p+1}} F^{ij} v_{;i} v_{;j} \left(f'^2 - \frac{2f'}{v} - f'' \right) + 2 \left(\frac{\vartheta'}{\vartheta} - \frac{\vartheta''}{\vartheta'} \right) f' \frac{p}{F^{p+1}} F^{ij} u_{;i} v_{;j} \\
&+ \frac{p}{F^{p+1}} F^{ij} z_{;i} z_{;j} - 2 \frac{p}{F^{p+1}} F^{ij} z_{;i} \left(f' v_{;j} + \left(\frac{\vartheta'}{\vartheta} - \frac{\vartheta''}{\vartheta'} \right) u_{;j} \right).
\end{aligned}
$$

Pick

$$
f(v) = -\log(v^{-\alpha} - \beta),
$$

where

$$
0 < \beta < \frac{1}{2v}, \quad 1 - \frac{\beta}{2} < \alpha < 1.
$$

Then

$$
1 - f'v = \frac{(1-\alpha) v^{-\alpha} - \beta}{v^{-\alpha} - \beta} \le \frac{\beta(\frac{v^{-\alpha}}{2} - 1)}{v^{-\alpha} - \beta} \le -\frac{\beta}{2} < 0
$$

and

$$
f'^2 - \frac{2}{v} f' - f'' = \frac{\alpha v^{-(\alpha+2)}}{v^{-\alpha} - \beta} (\alpha - 1) \le \frac{3}{4} \frac{\alpha - 1}{v^2} < 0.
$$

Hence at a maximal point of z there holds

$$\begin{aligned}
&\dot{z} - \frac{p}{F^{p+1}} F^{ij} z_{;ij} \\
\le\ & \frac{1}{F^p}\left(-(p+1)h_n^n + c\frac{\vartheta'}{\vartheta} + c\frac{\vartheta'^2}{\vartheta^2}(h_n^n)^{-1}\right) \\
+\ & \frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}F_k^k\left(-\frac{\beta}{2} + c_\epsilon\left(1+\frac{\vartheta'}{\vartheta}\kappa_n^{-1}\right)\|\nabla u\|^2 + c\frac{\vartheta'}{\vartheta}\frac{\|\nabla u\|}{\vartheta'^2}\kappa_n^{-1}\right) \\
+\ & \left(\frac{3}{4}\frac{\alpha-1}{v^2} + \epsilon c f'\right)\frac{p}{F^{p+1}} F^{ij} v_{;i} v_{;j},
\end{aligned}$$

where we used

$$2|F^{ij} u_{;i} v_{;j}| \le \frac{\epsilon\vartheta}{\vartheta'} F^{ij} v_{;i} v_{;j} + \frac{\vartheta'}{\epsilon\vartheta} F^{ij} u_{;i} u_{;j}$$

with sufficiently small ϵ. In case $\sup_{r>0}\vartheta'(r) = \infty$ or $p<1$, we have $\|\nabla u\|^2 \to 0$ and hence the result follows from the maximum principle. If $\vartheta' \le c$ and $p=1$ we supposed that

$$F = n\frac{H_{k+1}}{H_k}$$

which implies $F_k^k \le c$, cf. [49, Lemma 2.7], and hence

$$\frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}} F_k^k \le c\frac{\vartheta'}{\vartheta}\frac{1}{F^p}$$

due to Lemma 4.3. Hence the term $-(p+1)h_n^n$ dominates the whole evolution and we also obtain the bound on z in this case. $\square$

4.5. *Remark.* Lemma 4.4 is the only place where we need that F has this special form in case of bounded ϑ'. Of course the Euclidean case is excluded from this restriction, since the error terms involving $\|\nabla u\|^2$ will not appear here. However, the Euclidean case has already been settled in [22].

4.2. **Decay.** The global bounds from Lemma 4.3 and Lemma 4.4 as well as

$$F_{|\partial\Gamma} = 0$$

imply that the rescaled principal curvatures

$$\tilde{\kappa}_i = \frac{\vartheta}{\vartheta'}\kappa_i$$

range in a compact subset of Γ and hence the elliptic operator $d_h F$ is uniformly bounded,

$$c\|\xi\|^2 \le d_h F(\xi,\xi) \le C\|\xi\|^2.$$

The aim of this final section is to show that all $\tilde{\kappa}_i$ actually behave according to the convergence rates described in Theorem 1.3. The following two lemmata prepare this result. Throughout this whole section, the procedure is similar to the one in [60].

4.6. **Lemma.** *Under the assumptions of Theorem 1.3 (ii), along* (1.1) *there exist constants* μ *and* c*, such that*

$$\frac{\vartheta'}{\vartheta}\frac{1}{F} - \frac{1}{n} \le \frac{c}{\vartheta'^\mu}$$

Proof. We only have to consider the case that ϑ' is unbounded. Come back to the proof of Lemma 4.3 and consider (4.5) with

$$f(v) = \log v.$$

Hence from (4.5) we deduce that

$$z = \log\left(\frac{1}{F^p}\right) + \log v + p\log\vartheta' - p\log\vartheta + p\log n$$

satisfies

$$\begin{aligned}
&\dot z - \frac{p}{F^{p+1}}F^{ij}z_{;ij}\\
\le{}& \frac{p(p+1)}{F^{p+1}}\frac{\vartheta''}{\vartheta}\left(-F^{ij}g_{ij} + \frac{\vartheta}{\vartheta'}Fv^{-1} + c\|\nabla u\|^2\right)\\
&+ \frac{\vartheta'^2}{\vartheta^2}\frac{p}{F^{p+1}}\left((p-1)F^{ij}g_{ij} + \frac{1-p^2}{p}\frac{\vartheta}{\vartheta'}Fv^{-1} + \frac{\vartheta}{\vartheta'}\frac{p-1}{p}Fv + c\|\nabla u\|^2\right)\\
&+ \frac{p}{F^{p+1}}F^{ij}z_{;i}z_{;j} - \frac{2}{v}\frac{p}{F^{p+1}}F^{ij}z_{;i}v_{;j} - 2p\left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta}\right)\frac{p}{F^{p+1}}F^{ij}z_{;i}u_{;j}\\
\le{}& \frac{np(p+1)}{F^{p+1}}\frac{\vartheta''}{\vartheta}\left(-1 + e^{-\frac{z}{p}} + c\|\nabla u\|^2\right) + \frac{np(1-p)}{F^{p+1}}\frac{\vartheta'^2}{\vartheta^2}\left(-1 + e^{-\frac{z}{p}} + c\|\nabla u\|^2\right)\\
&+ \frac{p}{F^{p+1}}F^{ij}z_{;i}z_{;j} - \frac{2}{v}\frac{p}{F^{p+1}}F^{ij}z_{;i}v_{;j} - 2p\left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta}\right)\frac{p}{F^{p+1}}F^{ij}z_{;i}u_{;j}.
\end{aligned}$$

For $\mu \ge 0$ define

$$\rho = (e^z - 1)\vartheta'^{\mu}.$$

Then

$$\begin{aligned}
&\dot\rho - \frac{p}{F^{p+1}}F^{ij}\rho_{;ij}\\
={}& \left(\dot z - \frac{p}{F^{p+1}}z_{;ij}\right)e^z\vartheta'^{\mu} - \frac{p}{F^{p+1}}F^{ij}z_{;i}z_{;j}e^z\vartheta'^{\mu} + \mu\frac{\vartheta''}{\vartheta'}\left(\dot u - \frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j}\right)\rho\\
&- \left(\mu(\mu-1)\frac{\vartheta''^2}{\vartheta'^2} + \mu\frac{\vartheta'''}{\vartheta'}\right)\frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j}\rho\\
\le{}& \frac{np(p+1)}{F^{p+1}}\frac{\vartheta''}{\vartheta}e^{\frac{p-1}{p}z}\Big((1 - e^{\frac{z}{p}})\vartheta'^{\mu} + \frac{\mu}{np}F\frac{\vartheta}{\vartheta'}v^{-1}e^{\frac{1-p}{p}z}\rho\\
&\qquad - \frac{\mu}{n(p+1)}F^{ij}g_{ij}e^{\frac{1-p}{p}z}\rho + c_\mu\|\nabla u\|^2\vartheta'^{\mu}\Big)\\
&+ \frac{np(1-p)}{F^{p+1}}\frac{\vartheta'^2}{\vartheta^2}e^{\frac{p-1}{p}z}\left(-e^{\frac{z}{p}} + 1 + c\|\nabla u\|^2\right)\vartheta'^{\mu} - \frac{2}{v}\frac{p}{F^{p+1}}F^{ij}z_{;i}v_{;j}e^z\vartheta'^{\mu}\\
&- 2p\left(\frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta}\right)\frac{p}{F^{p+1}}F^{ij}z_{;i}u_{;j}e^z\vartheta'^{\mu}.
\end{aligned}$$

Now we estimate at maximal points of ρ and thus may assume $z > 0$. Then, also using

$$0 = \vartheta'^{-\mu}\rho_{;i} = z_{;i}e^z + \mu\frac{\vartheta''}{\vartheta'}(e^z - 1)u_{;i},$$

we obtain

$$\begin{aligned}
&\dot\rho - \frac{p}{F^{p+1}}F^{ij}\rho_{;ij}\\
&\le \frac{np(p+1)}{F^{p+1}}\frac{\vartheta''}{\vartheta}e^{\frac{p-1}{p}z}\Big(-\rho+\frac{\mu}{np}F\frac{\vartheta}{\vartheta'}v^{-1}e^{\frac{1-p}{p}z}\rho\\
&\qquad -\frac{\mu}{n(p+1)}F^{ij}g_{ij}e^{\frac{1-p}{p}z}\rho+c\|\nabla u\|^2\vartheta'^{\mu}\Big)\\
&\quad+\frac{np(1-p)}{F^{p+1}}\frac{\vartheta'^2}{\vartheta^2}e^{\frac{p-1}{p}z}\left(-\rho+c\|\nabla u\|^2\vartheta'^{\mu}\right),
\end{aligned}\tag{4.7}$$

which is negative for sufficiently small μ and large times, due to Lemma 3.3 and the remarks at the beginning of this section. The proof is complete. □

4.7. **Lemma.** *Under the assumptions of Theorem 1.3 (ii), along* (1.1) *the i-th rescaled principal curvature converges uniformly to* 1,

$$\left|v\kappa_i\frac{\vartheta}{\vartheta'}-1\right|\to 0,$$

provided ϑ' is unbounded.

Proof. Using (4.6) with $f(v)=\log v$ we obtain that

$$z=\log h^n_n+\log\frac{\vartheta}{\vartheta'}+\log v$$

satisfies

$$\begin{aligned}
&\dot z-\frac{p}{F^{p+1}}F^{ij}z_{;ij}\\
&\le \frac{p}{F^{p+1}}F^{ij}(\log h^n_n)_{;i}(\log h^n_n)_{;j}+c\frac{\vartheta'^2}{\vartheta^2}\frac{1}{F^{p+1}}\|\nabla u\|^2-\frac{p+1}{F^p}\frac{\vartheta'}{\vartheta}v^{-1}\left(e^z-1\right)\\
&\quad+\left(\frac{\vartheta''}{\vartheta'}-\frac{\vartheta'}{\vartheta}\right)\frac{p+1}{F^p}v^{-1}\left(e^{-z}-1\right)+\frac{\vartheta'}{\vartheta}\frac{p-1}{F^p}v\left(1-e^{-z}\right)+\frac{c}{F^{p+1}}\frac{1}{\vartheta^2}\|\nabla u\|\\
&=\frac{p}{F^{p+1}}F^{ij}(\log h^n_n)_{;i}(\log h^n_n)_{;j}-\frac{\vartheta''}{\vartheta'}\frac{p+1}{F^p}v^{-1}(1-e^{-z})+c\frac{\vartheta'^2}{\vartheta^2}\frac{1}{F^{p+1}}\|\nabla u\|^2\\
&\quad-\frac{\vartheta'}{\vartheta}\frac{p+1}{F^p}v^{-1}e^{-z}(e^z-1)^2+\frac{\vartheta'}{\vartheta}\frac{p-1}{F^p}v\left(1-e^{-z}\right)+\frac{c}{F^{p+1}}\frac{1}{\vartheta^2}\|\nabla u\|.
\end{aligned}$$

Define

$$\rho=(e^z-1)\vartheta'^{\mu},$$

with $\mu\ge 0$. ρ satisfies

$$\begin{aligned}
&\dot\rho-\frac{p}{F^{p+1}}F^{ij}\rho_{;ij}\\
&=\left(\dot z-\frac{p}{F^{p+1}}z_{;ij}\right)e^z\vartheta'^{\mu}-\frac{p}{F^{p+1}}F^{ij}z_{;i}z_{;j}e^z\vartheta'^{\mu}+\mu\frac{\vartheta''}{\vartheta'}\left(\dot u-\frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j}\right)\rho\\
&\quad-\left(\mu(\mu-1)\frac{\vartheta''^2}{\vartheta'^2}+\mu\frac{\vartheta'''}{\vartheta'}\right)\frac{p}{F^{p+1}}F^{ij}u_{;i}u_{;j}\rho\\
&\le -\frac{\vartheta''}{\vartheta'}\frac{p+1}{F^p}v^{-1}\rho-\frac{\vartheta'}{\vartheta}\frac{p+1}{F^p}v^{-1}(e^z-1)\rho-\frac{\vartheta'}{\vartheta}\frac{1-p}{F^p}v\rho+\frac{\vartheta'^2}{\vartheta^2}\frac{c}{F^{p+1}}\|\nabla u\|^2\vartheta'^{\mu}\\
&\quad+\frac{c}{F^{p+1}}\frac{1}{\vartheta^2}\|\nabla u\|\vartheta'^{\mu}+\frac{p}{F^{p+1}}F^{ij}(\log h^n_n)_{;i}(\log h^n_n)_{;j}e^z\vartheta'^{\mu}\\
&\quad-\frac{p}{F^{p+1}}F^{ij}z_{;i}z_{;j}e^z\vartheta'^{\mu}+\mu\frac{\vartheta''}{\vartheta'}\frac{p+1}{F^p}v^{-1}\rho-\mu\frac{\vartheta''}{\vartheta}\frac{p}{F^{p+1}}F^{ij}g_{ij}\rho.
\end{aligned}$$

At spatial maxima of ρ we have

$$\begin{aligned} 0 = \vartheta'^{-\mu}\rho_{;i} &= z_{;i}e^z + \mu\frac{\vartheta''}{\vartheta'}(e^z-1)u_{;i} \\ &= h^n_{n;i}v\frac{\vartheta}{\vartheta'} + h^n_n\left(\frac{\vartheta}{\vartheta'}\right)_{;i}v + h^n_n\frac{\vartheta}{\vartheta'}v_{;i} + \mu\frac{\vartheta''}{\vartheta'}(e^z-1)u_{;i} \end{aligned}$$

and hence

$$F^{ij}(\log h^n_n)_{;i}(\log h^n_n)_{;j} \le c\frac{\vartheta'^2}{\vartheta^2}\|\nabla u\|^2.$$

We obtain at a maximal point where $\rho > 0$

$$\begin{aligned} &\dot\rho - \frac{p}{F^{p+1}}F^{ij}\rho_{;ij} \\ \le &-\frac{\vartheta'}{\vartheta}\frac{p+1}{F^p}v^{-1}(e^z-1)\rho - \frac{\vartheta'}{\vartheta}\frac{1-p}{F^p}v\rho \\ &+\frac{n}{F^p}\frac{\vartheta''}{\vartheta'}v^{-1}\rho\left(-\mu p\frac{v}{F}\frac{\vartheta'}{\vartheta} + (p+1)\frac{\mu-1}{n}\right) + \frac{\vartheta'^2}{\vartheta^2}\frac{c}{F^{p+1}}\|\nabla u\|^2\vartheta'^\mu \\ &+\frac{c}{F^{p+1}}\frac{1}{\vartheta^2}\|\nabla u\|\vartheta'^\mu \\ \le &\frac{\vartheta'}{\vartheta}\frac{p+1}{vF^p}\left(-\frac{1-p}{p+1}\rho - (e^z-1)\rho + c\|\nabla u\|^2\vartheta'^\mu + \frac{c\vartheta'^\mu}{\vartheta'^2}\|\nabla u\|\right) \\ &+\frac{n}{F^p}\frac{\vartheta''}{\vartheta'}v^{-1}\rho\left(-\mu p\frac{v}{F}\frac{\vartheta'}{\vartheta} + (p+1)\frac{\mu-1}{n}\right). \end{aligned} \tag{4.8}$$

Set

$$\tilde\rho(t) = \max_M \rho(t,\cdot).$$

Note that $\tilde\rho$ is Lipschitz continuous, hence differentiable almost everywhere in $(0,\infty)$ and at points of differentiability there holds

$$\dot{\tilde\rho}(t) = \frac{\partial\rho}{\partial t}(t,x_t),$$

where

$$\rho(t,x_t) = \tilde\rho(t),$$

cf. [23, Lemma 6.3.2]. The original idea of this useful fact goes back to Hamilton [30, Lemma 3.5]. Choosing $\mu > 0$ small enough, $\|\nabla u\|^2\vartheta'^\mu$ converges to zero due to Lemma 3.3 and we obtain that for sufficiently large t,

$$\dot{\tilde\rho}(t) \le 0$$

on the set $\{\tilde\rho \ge 1\}$, provided $p < 1$. Hence in this case ρ is bounded. In case $p = 1$ we set $\mu = 0$ and obtain that for all $\epsilon > 0$ there exist $\delta_\epsilon > 0$ and T_ϵ, such that for all $t \ge T_\epsilon$ where $\tilde\rho$ is differentiable, there holds

$$\tilde\rho(t) \ge \epsilon \quad\Rightarrow\quad \dot{\tilde\rho}(t) < -\delta_\epsilon.$$

[58, Lemma 4.2] implies $\limsup_{t\to\infty}\tilde\rho \le 0$. Hence

$$\limsup_{t\to\infty} v\kappa_n\frac{\vartheta}{\vartheta'} \le 1$$

in both cases. Now

$$\sum_{i=1}^n \frac{\left(1 - v\kappa_i\frac{\vartheta}{\vartheta'}\right)}{nvF\frac{\vartheta}{\vartheta'}} = \frac{n - vH\frac{\vartheta}{\vartheta'}}{nvF\frac{\vartheta}{\vartheta'}} \le \frac{n - vF\frac{\vartheta}{\vartheta'}}{nvF\frac{\vartheta}{\vartheta'}} \le \frac{\vartheta'}{F\vartheta} - \frac{1}{n} \le c\vartheta'^{-\mu}$$

and hence

$$1 - v\kappa_1 \frac{\vartheta}{\vartheta'} \leq c\vartheta'^{-\mu} + \sum_{i=2}^{n} \left(v\kappa_i \frac{\vartheta}{\vartheta'} - 1 \right). \tag{4.9}$$

The proof is complete. □

Now we are in the position to optimise the decay estimates. We start with the gradient.

4.8. **Lemma.** *Under the assumption of Theorem 1.3 (ii) the function*

$$\tilde{z} = |\hat{\nabla}\varphi|^2 \vartheta'^{2p}$$

is uniformly bounded. If $p = 1$, then additionally there exist constants c and α such that

$$|\hat{\nabla}\varphi|^2 \leq ce^{-\alpha t}.$$

Proof. If ϑ' is unbounded, using Lemma 4.7 we can rewrite the evolution of

$$z = f(u)|\hat{\nabla}\varphi|^2$$

from (3.2) with

$$f(u) = \vartheta'^{\gamma}(u)$$

at maximal points as

$$\begin{aligned}
\mathcal{L}z &\leq \frac{2z}{F^{p+1}} \Big((p-1)\frac{\vartheta'}{\vartheta} vF - p\frac{\vartheta''}{\vartheta} F^k_k + \gamma \frac{p+1}{2} \frac{\vartheta''}{\vartheta'} \frac{F}{v} - \frac{\gamma np}{2} \frac{\vartheta''}{\vartheta} \Big) \\
&\quad - \frac{2f}{\vartheta^2} \frac{p}{F^{p+1}} F^k_l \tilde{g}^{lr} \hat{R}^m_{ikr} \varphi^i \varphi_m + \frac{p}{F^{p+1}} F^{kr} u_k u_r z \left(2\frac{f'^2}{f^2} - \frac{f''}{f} - \frac{f'}{f}\frac{\vartheta'}{\vartheta} \right) \\
&\leq \frac{2z}{F^{p+1}} \left(o(1) \frac{\vartheta'^2}{\vartheta^2} + n(p-1)\frac{\vartheta'^2}{\vartheta^2} - np\frac{\vartheta''}{\vartheta} + \frac{\gamma n}{2} \frac{\vartheta''}{\vartheta} \right) \\
&\quad - \frac{2f}{\vartheta^2} \frac{p}{F^{p+1}} F^k_l \tilde{g}^{lr} \hat{R}^m_{ikr} \varphi^i \varphi_m.
\end{aligned} \tag{4.10}$$

Since we want to bound z, it suffices to consider spatial maxima at which z is positive. At such there holds

$$\mathcal{L}z \leq \frac{2z}{F^{p+1}} \frac{\vartheta'^2}{\vartheta^2} \left(o(1) + n(p-1) - n\left(p - \frac{\gamma}{2}\right) \frac{\vartheta''\vartheta}{\vartheta'^2} - \frac{p}{\vartheta'^2} \widehat{\mathrm{Rc}} \left(\frac{\hat{\nabla}\varphi}{|\hat{\nabla}\varphi|}, \frac{\hat{\nabla}\varphi}{|\hat{\nabla}\varphi|} \right) \right). \tag{4.11}$$

In case $p < 1$ with $\gamma = 2p$, the right hand side is eventually negative for large t, since only the case of unbounded ϑ' has to be considered to prove the first statement. In case $p = 1$ we put $\gamma = 0$ and use the first estimate in (4.10) if ϑ' is bounded, whereas if ϑ' is unbounded we use (4.11), to get

$$\mathcal{L}z \leq -\delta z$$

for some δ and large times. The exponential decay follows. To prove the remaining claim, we evaluate (3.3) with

$$f = \vartheta'^2, \quad p = 1$$

and see

$$\mathcal{L}z \leq -\frac{2}{F^2} \frac{\vartheta''}{\vartheta} \left(1 + 1 - ce^{-\alpha t} - 2\right) F^k_k z + \frac{c}{F^2} \frac{\vartheta'^2}{\vartheta^2} e^{-\alpha t} z.$$

Hence the function

$$\bar{z}(t) = \max_{\mathcal{S}_0} z(t, \cdot)$$

satisfies

$$\dot{\bar{z}} \leq ce^{-\alpha t} \bar{z}$$

and is thus bounded. ∎

We optimise the convergence rate of the rescaled principal curvatures.

4.9. **Lemma.** *Under the assumptions of Theorem 1.3 (ii), along* (1.1) *there exists a constant c, such that for all $1 \leq i \leq n$, the i-th rescaled principal curvature satisfies*

$$\left| v\kappa_i \frac{\vartheta}{\vartheta'} - 1 \right| \leq \frac{ct}{\vartheta'^{p(p+1)}},$$

where we may drop the t-factor if $p < 1$ or if ϑ' is bounded.

Proof. Only the case that ϑ' is unbounded has to be considered.

(i) First we optimise the decay in Lemma 4.6. Using the optimal gradient estimates Lemma 4.8, we see from (4.7) that Lemma 4.6 holds with any $\mu < p(p+1)$, if c is allowed to depend on a lower bound of $p(p+1) - \mu$.

Now consider the function ρ defined in the proof of Lemma 4.7 and obtain from (4.8) with $\mu < p(p+1)$ at points where $\rho \geq 1$ that

$$\begin{aligned}
&\dot{\rho} - \frac{p}{F^{p+1}} F^{ij} \rho_{;ij} \\
&\leq \frac{\vartheta'}{\vartheta} \frac{p+1}{vF^p} \left(-\frac{1-p}{p+1} \rho + c\|\nabla u\|^2 \vartheta'^{\mu} + \frac{c\vartheta'^{\mu}}{\vartheta'^2} \|\nabla u\| \right) \\
&\quad + \frac{n}{F^p} \frac{\vartheta''}{\vartheta'} v^{-1} \rho \left(\frac{-p\mu}{n} + o(1) + (p+1)\frac{\mu - 1}{n} \right) \\
&= \frac{\vartheta'}{\vartheta} \frac{p+1}{vF^p} \left(-\frac{1-p}{p+1} \rho + c\|\nabla u\|^2 \vartheta'^{\mu} + \frac{c\vartheta'^{\mu}}{\vartheta'^2} \|\nabla u\| \right) \\
&\quad + \frac{1}{F^p} \frac{\vartheta''}{\vartheta'} v^{-1} \rho \left(\mu - (p+1) + o(1) \right) \\
&< 0
\end{aligned}$$

in case $p < 1$ for large times. In case $p = 1$ the right hand side of this inequality eventually decays exponentially and thus

$$\rho \leq c$$

in both cases. Hence for any $\mu < p(p+1)$ we have

$$v\kappa_n \frac{\vartheta}{\vartheta'} - 1 \leq \frac{c_\mu}{\vartheta'^{\mu}}.$$

Now putting $\mu = p(p+1)$ in (4.7) we see that the function ρ defined in the proof of Lemma 4.6 satisfies at positive maximal points with $\rho \geq 1$

$$\begin{aligned}
&\dot{\rho} - \frac{p}{F^{p+1}} F^{ij} \rho_{;ij} \\
&\leq \frac{np(p+1)}{F^{p+1}} \frac{\vartheta''}{\vartheta} e^{\frac{p-1}{p} z} \Big(-\rho + (p+1) \frac{F}{n} \frac{\vartheta}{\vartheta'} v^{-1} e^{\frac{1-p}{p} z} \rho - p\rho + c\|\nabla u\|^2 \vartheta'^{p(p+1)} \Big) \\
&\quad + \frac{np(1-p)}{F^{p+1}} \frac{\vartheta'^2}{\vartheta^2} e^{\frac{p-1}{p} z} \left(-\rho + c\|\nabla u\|^2 \vartheta'^{p(p+1)} \right) \\
&< \frac{np(p+1)}{F^{p+1}} \frac{\vartheta''}{\vartheta} e^{\frac{p-1}{p} z} \left(c\vartheta'^{-p} \rho + c\|\nabla u\|^2 \vartheta'^{p(p+1)} \right) \\
&\quad + \frac{np(1-p)}{F^{p+1}} \frac{\vartheta'^2}{\vartheta^2} e^{\frac{p-1}{p} z} \left(-\rho + c\|\nabla u\|^2 \vartheta'^{p(p+1)} \right).
\end{aligned}$$

In case $p < 1$ we use

$$\frac{\vartheta''}{\vartheta} \le c\frac{\vartheta'^2}{\vartheta^2}$$

to absorb every decaying term into $-\rho$ in the second line. In case $p = 1$ we use

$$\vartheta'^{-p}\rho \le c$$

to conclude

$$\dot\rho - \frac{p}{F^{p+1}}F^{ij}\rho_{;ij} \le c.$$

Hence we obtain

$$\frac{\vartheta'}{\vartheta}\frac{1}{F} - \frac{1}{n} \le \frac{ct}{\vartheta'^{p(p+1)}},$$

and the same without the t-factor in case $p < 1$.

(ii) In the second step we optimise the convergence rate in Lemma 4.7. Therefore we consider the function ρ defined in that proof and obtain from (4.8) with $\mu = p(p+1)$ at points where $\rho \ge 1$ that

$$\begin{aligned}
&\dot\rho - \frac{p}{F^{p+1}}F^{ij}\rho_{;ij}\\
&\le \frac{\vartheta'}{\vartheta}\frac{p+1}{vF^p}\left(-\frac{1-p}{p+1}\rho + c\|\nabla u\|^2\vartheta'^{p(p+1)} + \frac{c\vartheta'^{p(p+1)}}{\vartheta'^2}\|\nabla u\|\right)\\
&\quad + \frac{n}{F^p}\frac{\vartheta''}{\vartheta'}v^{-1}\rho\left(\frac{-p^2(p+1)}{n} + o(1) + (p+1)\frac{p(p+1)-1}{n}\right)\\
&= \frac{\vartheta'}{\vartheta}\frac{p+1}{vF^p}\left(-\frac{1-p}{p+1}\rho + c\|\nabla u\|^2\vartheta'^{p(p+1)} + \frac{c\vartheta'^{p(p+1)}}{\vartheta'^2}\|\nabla u\|\right)\\
&\quad + \frac{1}{F^p}\frac{\vartheta''}{\vartheta'}v^{-1}\rho\left(p^2 - 1 + o(1)\right)\\
&\le 0
\end{aligned}$$

in case $p < 1$ for large times. In case $p = 1$ the right hand side of this inequality is bounded and thus

$$\rho \le ct$$

in this case. Estimating (4.9) with the optimised bounds completes the proof. $\square$

We finish the proof of Theorem 1.3 by proving the final statement about the exponential decay in item (ii). The function

$$z = \log\vartheta(u) - \frac{t}{n}$$

defined on $[0,\infty)\times\mathcal{S}_0$ satisfies

$$\dot z = \frac{v\vartheta'}{\vartheta F(\mathcal{W})} - \frac{1}{n} = \frac{v}{F\left(\frac{1}{v}\delta^i_j + \frac{1}{v^3\vartheta^2}u^iu_j - \frac{1}{v\vartheta'\vartheta}\tilde g^{ik}u_{kj}\right)} - \frac{1}{n} = G(y, z, \hat\nabla z, \hat\nabla^2 z).$$

Hence

$$\frac{\partial G}{\partial z_{ij}} = \frac{\vartheta'^{-2}}{F^2\left(\frac{\vartheta}{\vartheta'}\mathcal{W}\right)}F^i_k\tilde g^{kj},$$

which is uniformly elliptic, since ϑ' is globally bounded. z is uniformly bounded, as can be seen similarly as in [60, Prop. 3.1, Lemma 3.2]. Furthermore

$$|\hat\nabla z| \le c|\hat\nabla\varphi| \le ce^{-\alpha t}$$

and

$$|\hat\nabla^2\varphi| \le c.$$

Applying the regularity results of Krylov and Safonov as well as Schauder theory, we obtain uniform C^m-bounds for z. Due to interpolation we get

$$|\hat{\nabla}^2\varphi| \leq ce^{-\alpha t},$$

which implies (1.3) with t replaced by $e^{-\alpha t}$.

4.10. *Remark.* The previous argument is precisely the way to deduce a uniform bound on the rescaled principal curvatures for the inverse mean curvature flow, when ϑ' is bounded, as it was performed in [52, 71]. The crucial point is here, that one does not need curvature estimates to have F^{ij} uniformly elliptic. One only needs a bound on the rescaled speed

$$\tilde{H} = \frac{\vartheta}{\vartheta'}H.$$

Then the above argumentation applies.

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Appendix A5

LOCALLY CONSTRAINED INVERSE CURVATURE FLOWS

by Julian Scheuer and Chao Xia
submitted for publication

LOCALLY CONSTRAINED INVERSE CURVATURE FLOWS

JULIAN SCHEUER AND CHAO XIA

Abstract. We consider inverse curvature flows in warped product manifolds, which are constrained subject to local terms of lower order, namely the radial coordinate and the generalized support function. Under various assumptions we prove longtime existence and smooth convergence to a coordinate slice. We apply this result to deduce a new Minkowski type inequality in the anti-de-Sitter Schwarzschild manifolds and a weighted isoperimetric type inequality in the hyperbolic space.

Contents

1. Introduction

In this paper we deduce convergence results for hypersurface flows in $(n+1)$-dimensional warped product spaces

$$N^{n+1} = (a, b) \times \mathbb{S}^n.$$

The metric on N is supposed to have the form

$$\bar{g} = dr^2 + \lambda^2(r)\sigma,$$

where λ is a positive warping factor and σ is the round metric on $\mathbb{S}^n$. Precisely, let $M = M^n$ be a closed, connected and orientable smooth manifold, then for a family of embeddings

$$x\colon [0, T^*) \times M \to N,$$

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which satisfy the flow equation

$$\dot{x} = \left(\frac{n}{F} - \frac{u}{\lambda'(r)}\right)\nu \tag{1.1}$$
$$x(0,\cdot) = x_0,$$

we will prove long time existence and smooth convergence to a slice $\{r = \text{const}\}$. F is a function of the principal curvatures satisfying several natural properties to be specified later, u is the *support function*

$$u = \bar{g}(\lambda(r)\partial_r, \nu) \tag{1.2}$$

and x_0 is an initial embedding of M, the image of which is a graph over $\mathbb{S}^n$,

$$M_0 = x_0(M) = \{(r_0(y), y) \colon y \in \mathbb{S}^n\}.$$

Before we state the main results in detail, cf. Theorem 1.1, Theorem 1.3 and Theorem 1.5, let us give a brief overview over recent related work and our motivation to consider this flow.

Curvature driven hypersurface flows have attracted a lot of attention for about the last four decades, starting with the mean curvature flow, [5, 29, 30], and several fully nonlinear (1-homogeneous) analogues involving the scalar curvature, the Gaussian curvature and more general functions of the principal curvatures, [2, 3, 10, 11]. Beside these contracting flows also expanding flows for star-shaped hypersurfaces have been considered, [17, 21, 22, 23, 39, 40, 42]. The most prominent example of an expanding flow is the inverse mean curvature flow, a weak notion of which was used by Huisken and Ilmanen to prove the Riemannian Penrose inequality, [32]. Various other applications of contracting and expanding flows include a classification of 2-convex n-dimensional hypersurfaces using the mean curvature flow with surgery, due to Huisken and Sinestrari for $n \geq 3$, [34], various extensions of geometric inequalities of Alexandrov-Fenchel-type to non-convex hypersurfaces, [8], [25], new Alexandrov-Fenchel-type inequalities in the hyperbolic space [12, 15, 44, 45] and in the sphere [24, 36, 45].

These contracting and expanding flows all have the property of some sort of singularity formation, where however, in the optimal case, the singularities in the expanding case are quite easy to deal with and only manifest themselves in a uniform convergence to infinity or to a minimal hypersurface, if present. Still it seems tempting to directly define a flow which prevents this singularity formation, for example by adding a constraining term. The first example of such flows is the volume preserving mean curvature flow which has the form

$$\dot{x} = \left(\frac{1}{|M_t|}\int_{M_t} H - H\right)\nu. \tag{1.3}$$

It has the nice property that additionally to keeping the enclosed volume fixed it also decreases the surface area, making it a natural candidate to prove the isoperimetric inequality, once one can show that it drives hypersurfaces to round spheres. In [31] this was accomplished for strictly convex hypersurfaces of the Euclidean space. Similar flows, which preserve higher order curvature integrals, where considered for example in [37, 38] and in [9] for flows in the hyperbolic space. Note however that the global term involved in this equation adds such heavy complications, that these nonlocal flows until now only allowed a quite restricted class of hypersurfaces, namely convex ones in the Euclidean space and horo-convex[1] ones in the hyperbolic space. Beside some perturbation results, in the sphere there are even no results at all, [1].

[1] A hypersurface in the hyperbolic space is called horo-convex if all its principal curvatures are greater or equal than 1.

However, using the Minkowski identity in $\mathbb{R}^{n+1}$,

$$\int_M H \langle x, \nu \rangle = n|M|,$$

it is possible to define a constrained flow, which involves no global term and still preserves enclosed volume while decreasing the surface area. In the Euclidean space it reads

$$\dot{x} = (n - H \langle x, \nu \rangle)\nu$$

and in warped products as above with warping factor $\lambda(r)$ it has to be

$$\dot{x} = (n\lambda'(r) - Hu)\nu, \tag{1.4}$$

where u is defined as (1.2). This beautiful flow was invented by Guan and Li in [26], where they proved longtime existence and smooth convergence to a round sphere when the ambient space is a space form. Together with Mu-Tao Wang they generalized this result to a broader class of ambient warped products with mild assumptions on λ in [28]. The major advantage compared to the classical volume preserving mean curvature flow (1.3) is that the C^0-estimates a.k.a. barriers are for free due to the maximum principle. Hence only the starshapedness of the initial hypersurface is required, namely that it is a graph in the warped product $(a, b) \times \mathbb{S}^n$ over the base $\mathbb{S}^n$. This result allows to deduce an isoperimetric inequality for such graphs in quite general warped products. See also [27] for a fully nonlinear extension of this flow.

On the other hand, Brendle, Guan and Li [7] designed an inverse type constrained curvature flow in space forms,

$$\dot{x} = \left(\frac{n\lambda'}{F} - u\right)\nu. \tag{1.5}$$

Compared to the mean curvature type constrained flow (1.4), this flow seems more appropriate for higher order isoperimetric type inequalities – the Alexandrov-Fenchel type inequalities for quermassintegrals – in space forms, for the reason that the higher order Minkowski identities imply that for

$$F = n\frac{H_k}{H_{k-1}}$$

the k-th quermassintegral is preserved, while the $(k+1)$-th quermassintegral is decreasing. However, the study of (1.5) is quite subtle from the PDE point of view and until today no satisfactory complete result has been achieved. Some convergence results are proved in [7] when the initial hypersurface is already close to a sphere. A full convergence result for closed, starshaped and k-convex initial hypersurfaces would prove the quermass Alexandrov-Fenchel inequalities for such hypersurfaces. For horo-convex domains these have been established by Wang and the second author [44] using a global quermassintegral preserving curvature flow.

Guan-Li's considerations motivate us to study another kind of constrained flow, the constrained inverse curvature flow (1.1) in general warped product spaces. Compared to (1.5), we are able to prove the longtime existence and smooth convergence of (1.1) to a coordinate slice under mild assumptions on the curvature function F, the warping factor λ and the initial hypersurface. We use this result to deduce a new geometric inequality in the anti-de-Sitter Schwarzschild manifolds, cf. Theorem 1.5, on which we will give more comments later.

Let us first state the main results of this paper. Since our assumptions on the curvature function and the initial embedding depend on the structure of the warping factor λ, we split our flow results into two theorems. We start with the ambient space $N = \mathbb{S}^{n+1}_+$, in which case $\lambda(r) = \sin r, r \in [0, \frac{\pi}{2})$.

1.1. **Theorem.** *Let $x_0(M)$ be the embedding of a closed n-dimensional manifold M into $\mathbb{S}^{n+1}$, such that $x_0(M)$ is strictly convex. Let*

$$F = n\frac{H_k}{H_{k-1}},$$

where H_k is the k-th normalized elementary symmetric polynomial of the principal curvatures. Then any solution x of (1.1) *exists for all positive times and converges to a geodesic slice in the C^∞-topology.*

Now we come to ambient spaces satisfying $\lambda'' \geq 0$. We obtain convergence results for a large class of speeds and therefore make the following assumption.

1.2. **Assumption.** Let $\Gamma \subset \mathbb{R}^n$ be a symmetric, convex, open cone containing

$$\Gamma_+ = \{(\kappa_i) \in \mathbb{R}^n \colon \kappa_i > 0\}$$

and suppose that F is positive in Γ, strictly monotone, homogeneous of degree one and concave with

$$F_{|\partial\Gamma} = 0, \quad F(1,\dots,1) = n.$$

1.3. **Theorem.** *Let $a, b \in \mathbb{R}$ and $(N, \bar{g})$ be the warped space $((a,b) \times \mathbb{S}^n, dr^2 + \lambda^2(r)\sigma)$ with $\lambda > 0$, $\lambda' > 0$ and $\lambda'' \geq 0$. Let $F \in C^\infty(\Gamma)$ satisfy Assumption 1.2 and let $x_0(M)$ be the embedding of a closed n-dimensional manifold M into N, such that $x_0(M)$ is a graph over the domain $\mathbb{S}^n$ and such that $\kappa \in \Gamma$ for all n-tuples of principal curvatures along $x_0(M)$. Then any solution x of* (1.1) *exists for all positive times and converges to a geodesic slice in the C^∞-topology.*

1.4. *Remark.* The assumption $\lambda'' \geq 0$ is only used for deriving the uniform lower bound for F. This assumption resembles the non-positivity of the ambient sectional curvature in the radial direction, a property which was also crucial in the deduction of long-time existence of the inverse mean curvature flow in warped product spaces, cf. [40].

Note that compared to the purely expanding inverse mean curvature flow

$$\dot{x} = \frac{1}{H}\nu, \tag{1.6}$$

which was treated in general warped products in [40], the set of assumptions on the warping factor in Theorem 1.3 is quite small. In order to obtain convergence results of a purely expanding flow, ones needs a lot of more global information about the ambient space. From the viewpoint of geometric inequalities for hypersurfaces, only local information is required and hence a constrained flow seems to be more promising than a flow of the form (1.6). Indeed, in this paper, we use Theorem 1.3 to obtain the following geometric inequalities, one weighted Minkowski-type inequality and one weighted isoperimetric type inequality.

1.5. **Theorem.** *Let $N = (a,b) \times \mathbb{S}^n$ be equipped with one of the anti-de-Sitter Schwarzschild metrics or the hyperbolic metric, i.e.*

$$\lambda' = \sqrt{1 + \lambda^2 - m\lambda^{1-n}}, \quad m \geq 0.$$

Let $\Sigma \subset N$ be a closed, star-shaped and mean-convex hypersurface, given by the function $r \colon \mathbb{S}^n \to (a,b)$, and let

$$\Omega = \{(s,y) \in N \colon a \leq s \leq r(y), \quad y \in \mathbb{S}^n\}.$$

Then there hold

$$\int_\Sigma H\lambda' d\mu - 2n\int_\Omega \frac{\lambda'\lambda''}{\lambda} dN \geq \xi_1(|\Sigma|) \tag{1.7}$$

and

$$\int_\Sigma H\lambda' d\mu - 2n\int_\Omega \frac{\lambda'\lambda''}{\lambda} dN \geq \xi_0\left(\int_\Omega \lambda' dN\right), \tag{1.8}$$

where ξ_0, ξ_1 *are the associated monotonically increasing functions for radial coordinate slices. Equality holds if and only if* Σ *is a radial coordinate slice.*

In particular, in the hyperbolic space, due to $\lambda'' = \lambda$, inequality (1.8) reduces to

$$\int_\Sigma H\lambda' d\mu - (n+1)n\int_\Omega \lambda' dN \geq n|\mathbb{S}^n|^{\frac{2}{n+1}}\left((n+1)\int_\Omega \lambda' dN\right)^{\frac{n-1}{n+1}}, \tag{1.9}$$

where $\lambda'(r) = \cosh r$. Equality in (1.9) holds if and only if Σ is a geodesic sphere centred at the origin. The second author proved a Minkowski type inequality in [46] stating that for a closed horo-convex hypersurface $\Sigma \subset \mathbb{H}^{n+1}$ there holds

$$\left(\int_\Sigma \lambda' d\mu\right)^2 \geq \frac{n+1}{n}\int_\Sigma H\lambda' d\mu \int_\Omega \lambda' dN.$$

Combining this with (1.9), we get:

1.6. **Theorem.** *Let* Σ *be a closed horo-convex hypersurface in* $\mathbb{H}^{n+1}$ *with the origin lying inside* Ω. *Then*

$$\int_\Sigma \lambda' d\mu \geq \left[\left((n+1)\int_\Omega \lambda' dN\right)^2 + |\mathbb{S}^n|^{\frac{2}{n+1}}\left((n+1)\int_\Omega \lambda' dN\right)^{\frac{2n}{n+1}}\right]^{\frac{1}{2}}.$$

Equality holds if and only if Σ *is a geodesic sphere centred at the origin.*

1.7. *Remark.* Theorem 1.6 already appeared in the paper [16], where it is the case $k = 0$ in Thm. 9.2. However, their proof relies on an *invalid* inequality, namely [16, equ. (9.8)], which states

$$|\Sigma|^{\frac{n+1}{n}} \geq |\mathbb{S}^n|^{\frac{1}{n}}\int_\Sigma u d\mu \quad \left(= |\mathbb{S}^n|^{\frac{1}{n}}(n+1)\int_\Omega \lambda' dN\right).$$

This inequality is already incorrect on geodesic spheres not centred at the origin. Theorem 1.6 fixes this gap in the proof of [16, Thm. 9.2].

1.8. *Remark.* By using the classical inverse mean curvature flow, Brendle-Hung-Wang proved in [8] for a closed, star-shaped and mean-convex hypersurface Σ in anti-de-Sitter Schwarzschild space, that

$$\int_\Sigma H\lambda' d\mu - (n+1)n\int_\Omega \lambda' dN \geq n|\mathbb{S}^n|^{\frac{1}{n}}\left(|\Sigma|^{\frac{n-1}{n}} - |\partial N|^{\frac{n-1}{n}}\right). \tag{1.10}$$

In particular, in the hyperbolic space, they get

$$\int_\Sigma H\lambda' d\mu - (n+1)n\int_\Omega \lambda' dN \geq n|\mathbb{S}^n|^{\frac{1}{n}}|\Sigma|^{\frac{n-1}{n}}.$$

(1.10) is different from (1.7), in the sense that the right hand side of (1.7) does not depend on the horizon $\{a\} \times \mathbb{S}^n$.

Another nice corollary is given by the following area bound for star-shaped and mean convex hypersurfaces in ambient spaces of non-positive radial curvature. It is neither clear to the authors, whether this bound is evident by other means, nor if it has been recorded before. It follows from Theorem 1.3, the monotonicity of area in these spaces, cf. (8.2), and Remark 4.2.

1.9. **Corollary.** *Let $a, b \in \mathbb{R}$ and $(N, \bar{g})$ be the warped space $((a,b)\times \mathbb{S}^n, dr^2 + \lambda^2(r)\sigma)$ with $\lambda > 0$, $\lambda' > 0$ and $\lambda'' \geq 0$. Let $\Sigma \subset N$ be a closed, star-shaped and mean-convex hypersurface,*

$$\Sigma = \{(r(y), y) \in N \colon y \in \mathbb{S}^n\}.$$

Then the area of Σ satisfies

$$|\Sigma| \leq |\mathbb{S}^n| \lambda\left(r_{max}\right)^n,$$

where $r_{max} = \max_{\mathbb{S}^n} r$.

It would be very interesting to find further monotone quantities along these flows, in particular in a spherical ambient space.

The paper is organised as follows. In sections 2 and 3, we collect the notation and derive the fundamental evolution equations for several geometric quantities. In sections 4 to 7, we derive a priori estimates under various conditions on F and λ and in section 8 we complete the proof of Theorem 1.1 and Theorem 1.3. Section 9 is devoted to prove monotonicity for various geometric quantities and in turn the geometric inequalities in Theorem 1.5.

2. Notation and conventions

2.1. Conventions on Riemannian geometry.

Intrinsic Curvature. Let (M^n, g) be a Riemannian manifold. With respect to a local frame $(e_i)_{1\leq i\leq n}$ of the tangent bundle, let (g_{ij}) denote the coordinate functions of g with respect to the basis $(\epsilon^i \otimes \epsilon^j)_{1\leq i,j\leq n}$, where ϵ^i denote the basis elements dual to e_i. Let (g^{ij}) denote the inverse matrix of (g_{ij}). For a (k,l)-tensor field T, the coordinates of which with respect to this frame are given by

$$T = (T^{i_1\dots i_k}_{j_1\dots j_l}),$$

we can define $(k+1, l-1)$-tensor fields by using the tangent-cotangent isomorphism induced by g, e.g.

$$T^{i_1\dots i_{k+1}}_{j_1\dots j_{l-1}} = T^{i_1\dots i_k}_{j_1\dots j_l} g^{j_l i_{k+1}}.$$

Of course we can also raise other indices to different slots, but it will always be apparent, or explicitly stated, which one is meant.

The Lie-Bracket of two vector fields X, Y on M is given by

$$[X,Y]\varphi = X(Y\varphi) - Y(X\varphi) \quad \forall \varphi \in C^\infty(M).$$

Let ∇ be the Levi-Civita connection of g, then for a (k,l) tensor field T, its covariant derivative ∇T is a $(k, l+1)$ tensor field given by

$$\begin{aligned}
&(\nabla T)(Y^1,\dots,Y^k,X_1,\dots,X_l,X)\\
&= (\nabla_X T)(Y^1,\dots,Y^k,X_1,\dots,X_l)\\
&= X(T(Y^1,\dots,Y^k,X_1,\dots,X_l)) - T(\nabla_X Y^1, Y^2,\dots,Y^k,X_1,\dots,X_l) - \dots\\
&\quad - T(Y^1,\dots,Y^k,X_1,\dots,X_{l-1}\nabla_X X_l).
\end{aligned}$$

We denote by $\nabla^m T$ the m-th covariant derivative of T and its coordinates with respect to a basis $(e_i)_{1\leq i\leq n}$ are denoted by

$$\nabla^m T = \left(T^{i_1\dots i_k}_{j_1\dots j_l; j_{l+1}\dots j_{l+m}}\right),$$

where all indices appearing after the semicolon indicate covariant derivatives. The $(1,3)$ Riemannian curvature tensor is defined by

$$\mathrm{Rm}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{2.1}$$

or with respect to the basis (e_i),

$$\mathrm{Rm}(e_i, e_j)e_k = R_{ijk}{}^l e_l,$$

where we use the summation convention (and will henceforth do so). The coordinate expression of (2.1), the so-called *Ricci-identities*, read

$$X^k{}_{;ij} - X^k{}_{;ji} = -R_{ijm}{}^k X^m \tag{2.2}$$

for all vector fields $X = (X^k)$. We also denote the $(0,4)$ version of the curvature tensor by Rm,

$$\mathrm{Rm}(W, X, Y, Z) = g(\mathrm{Rm}(W, X)Y, Z).$$

The Ricci curvature can unambiguously defined in coordinates by

$$\mathrm{Rc}(e_i, e_j) = R_{ij} = R_{kij}{}^k.$$

The scalar curvature is

$$R = R_i{}^i = g^{ki}R_{ki}.$$

Extrinsic curvature. When dealing with immersed hypersurfaces

$$x \colon M \hookrightarrow N$$

of a Riemannian manifold M^n into an ambient Riemannian manifold N^{n+1}, we furnish all the previous geometric quantities of N with an overbar, e.g. $\bar{g} = (\bar{g}_{\alpha\beta})$, where greek indices run from 0 to n, $\bar{\nabla}$ etc. We keep using latin indices, running from 1 to n, for geometric quantities of M, e.g. the induced metric $g = (g_{ij})$. The induced geometry of M is governed by the following relations. The (local) second fundamental form $h = (h_{ij})$ is given by the Gaussian formula

$$\bar{\nabla}_X Y = \nabla_X Y - h(X, Y)\nu, \tag{2.3}$$

where ν is a local normal field. Note that here (and for the rest of the paper), we will abuse notation by disregarding the necessity to distinguish between a vector $X \in T_pM$ and its push-forward $x_*X \in T_pN$. The Weingarten endomorphism $A = (h^i_j)$ is given by $h^i_j = g^{ki}h_{kj}$ and there holds the Weingarten equation

$$\bar{\nabla}_X \nu = A(X), \tag{2.4}$$

or in coordinates

$$\nu^\alpha{}_{;i} = h^k_i x^\alpha{}_{;k}.$$

We also have the Codazzi equation

$$\nabla_Z h(X, Y) - \nabla_Y h(X, Z) = -\overline{\mathrm{Rm}}(\nu, X, Y, Z), \tag{2.5}$$

or

$$h_{ij;k} - h_{ik;j} = -\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta{}_{;i} x^\gamma{}_{;j} x^\delta{}_{;k},$$

and the Gauss equation

$$\mathrm{Rm}(W, X, Y, Z) = \overline{\mathrm{Rm}}(W, X, Y, Z) + h(W, Z)h(X, Y) - h(W, Y)h(X, Z) \tag{2.6}$$

or

$$R_{ijkl} = \bar{R}_{\alpha\beta\gamma\delta} x^\alpha{}_{;i} x^\beta{}_{;j} x^\gamma{}_{;k} x^\delta{}_{;l} + h_{il}h_{jk} - h_{ik}h_{jl}.$$

Graphs in warped products. In this paper we deal with warped products

$$N = (a,b) \times \mathbb{S}^n$$

with metric

$$\bar{g} = dr^2 + \lambda^2(r)\sigma,$$

where σ is the round metric of $\mathbb{S}^n$. We need the specific structure of the Ricci curvature tensor in such a warped product. There holds

$$\overline{\mathrm{Rc}} = -\left(\frac{\lambda''}{\lambda} - (n-1)\frac{1-\lambda'^2}{\lambda^2}\right)\bar{g} - (n-1)\left(\frac{\lambda''}{\lambda} + \frac{1-\lambda'^2}{\lambda^2}\right) dr \otimes dr, \tag{2.7}$$

cf. [6, Prop. 2.1].

Our hypersurfaces

$$x \colon M \hookrightarrow N$$

will all be graphs over $\mathbb{S}^n$,

$$x(M) = \{(r(y), y) \colon y \in \mathbb{S}^n\} = \{(r(y(\xi)), y(\xi)) \colon \xi \in M\},$$

where we do not make a notational difference between the radial coordinate r of N and the function $r_{|M}$. Along M we will always pick the *outward* pointing normal

$$\nu = v^{-1}(1, -\lambda^{-2}\sigma^{ik}\partial_k r),$$

where

$$v^2 = 1 + \lambda^{-2}\sigma^{ij}\partial_i r \partial_j r,$$

and use this normal in the Gaussian formula (2.3). The support function of M is defined by

$$u = \bar{g}(\lambda\partial_r, \nu) = \frac{\lambda}{v}.$$

There is also a relation between the second fundamental form and the radial function on the hypersurface. Let

$$\bar{h} = \lambda'\lambda\sigma,$$

then there holds

$$v^{-1}h = -\nabla^2 r + \bar{h},$$

cf. [20, equ. (1.5.10)]. Since the induced metric is given by

$$g_{ij} = r_{;i}r_{;j} + \lambda^2\sigma_{ij},$$

we obtain

$$v^{-1}h_{ij} = -r_{;ij} + \frac{\lambda'}{\lambda}g_{ij} - \frac{\lambda'}{\lambda}r_{;i}r_{;j}. \tag{2.8}$$

Define

$$\varphi(r) = \int_a^r \frac{1}{\lambda}.$$

Regarding r as a function on $\mathbb{S}^n$, we have

$$h_i^j = \frac{\lambda'}{\lambda v}\delta_i^j - \frac{1}{\lambda v}\tilde{g}^{jk}\varphi_{,ki},$$

where

$$\tilde{g}^{ij} = \sigma^{ij} - \frac{\varphi_{,}{}^{i}\varphi_{,}{}^{j}}{v^2}$$

and the covariant derivative and index raising is performed with respect to the spherical metric σ_{ij}, cf. [21, equ. (3.26)]. We will use $\hat{\nabla}$ to denote the covariant derivative on $\mathbb{S}^n$ throughout this paper.

Anti-de-Sitter Schwarzschild space. The anti-de-Sitter Schwarzschild manifolds are asymptotically hyperbolic Riemannian warped products of the form

$$N = (r_0, \infty) \times \mathbb{S}^n$$

equipped with the warped product metric

$$\bar{g} = dr^2 + \lambda^2(r)\sigma,$$

where λ satisfies

$$\lambda' = \sqrt{1 + \lambda^2 - m\lambda^{1-n}}$$

with $m > 0$ and horizon $\partial N = \{r_0\} \times \mathbb{S}^n$. The limiting case $m = 0$ is the hyperbolic metric. These Riemannian manifolds carry the property to be *static*, i.e.

$$\bar{\Delta}\lambda'\bar{g} - \bar{\nabla}^2\lambda' + \lambda'\overline{\mathrm{Rc}} = 0,$$

which ensures that the Lorentzian warped product $-\lambda'^2 dt^2 + \bar{g}$ is a solution to Einstein's equation.

2.2. **Curvature functions.** In Assumption 1.2, the part of our normal variation that depends on the curvature of the hypersurface was stipulated to depend on the principal curvatures

$$F = F(\kappa_i).$$

However, in the calculation of the evolution equations it is often useful to consider F as a function of the diagonalisable Weingarten operator A,

$$F = F(A) := F(\mathrm{EV}(A)),$$

where $\mathrm{EV}(A)$ is the unordered n-tuple of eigenvalues of A. This is well-defined due to the symmetry of F. However, when using this definition, F is not defined on the whole endomorphism bundle, but only on the diagonalisable operators. It is thus most convenient to consider the function defined by,

$$\hat{F}(g,h) := F\left(\frac{1}{2}g^{ik}(h_{kj} + h_{jk})\right)$$

for all positive definite g and all bilinear forms $h \in T^{0,2}_p M$. Then

$$\hat{F}^{ij} = \frac{\partial F}{\partial h_{ij}}$$

is a $(2,0)$-tensor and we also write

$$\hat{F}^{ij,kl} = \frac{\partial F}{\partial h_{ij}\partial h_{kl}}.$$

Furthermore, if $F = F(\kappa_i)$ is strictly monotone, then $\hat{F}^{ij}$ is strictly elliptic. If F is concave, then

$$\hat{F}^{ij,kl}\eta_{ij}\eta_{kl} \leq 0$$

for all symmetric (η_{ij}). We refer to [4], [20, Ch. 2] and [41] for more details on curvature functions.

Furthermore we will abuse notation and also write F for $\hat{F}$, since no confusion will be possible. E.g., when writing F^{ij}, we can only mean $\hat{F}^{ij}$, since there are two contravariant indices.

Let us denote by σ_k the k-th elementary symmetric polynomial and define the k-th normalized elementary symmetric polynomial by

$$H_k = \frac{1}{\binom{n}{k}}\sigma_k.$$

Denote by Γ_k the connected component of $\{\sigma_k > 0\}$ which contains the point $(1,\dots,1)$.

3. Evolution equations

In this section we deduce the evolution equations relevant to study the flow

$$\dot{x} = \left(\frac{n}{F} - \frac{u}{\lambda'}\right)\nu \equiv \mathcal{F}\nu. \tag{3.1}$$

The following basic evolution equations are well known and can be found in many places. We use the reference [20, Ch. 2.3], where we note that we use the other sign on the curvature tensor.

3.1. **Lemma.** *Along* (3.1) *the following evolution equations hold:*

$$\dot{g} = 2\mathcal{F}h, \qquad \frac{\bar\nabla}{dt}\nu = -\operatorname{grad}\mathcal{F},$$

$$\dot{h}_i^j = -\mathcal{F}_{;i}{}^{j} - \mathcal{F}h_k^j h_i^k - \mathcal{F}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;i}\nu^\beta\nu^\gamma x^\delta_{;k}g^{kj}, \tag{3.2}$$

and

$$\dot{h}_{ij} = -\mathcal{F}_{;ij} + \mathcal{F}h_{ik}h_j^k - \mathcal{F}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;i}\nu^\beta\nu^\gamma x^\delta_{;j}. \tag{3.3}$$

We need some further special evolution equations.

3.2. **Lemma.** *Define the operator $\mathcal{L}$ by*

$$\mathcal{L} = \partial_t - \frac{n}{F^2}F^{ij}\nabla^2_{ij} - \frac{\lambda}{\lambda'}r_;{}^k\nabla_k.$$

Along the flow (3.1) *of graphs*

$$M_t = \{(r(t,y),y)\colon y\in\mathbb{S}^n\}$$

we have the following evolution equations for the radial function r, the support function u and the curvature function F:

$$\mathcal{L}r = \frac{2n}{vF} - \frac{\lambda}{\lambda'} - \frac{n\lambda'}{\lambda F^2}F^{ij}g_{ij} + \frac{n\lambda'}{\lambda F^2}F^{ij}r_{;i}r_{;j}, \tag{3.4}$$

$$\begin{aligned}\mathcal{L}u &= \frac{n}{F^2}\left(F^{ij}h_{ik}h_j^k - \frac{1}{n}F^2\right)u - \frac{\lambda''\lambda}{\lambda'^2}\|\nabla r\|^2 u\\ &\quad + \frac{n\lambda}{F^2}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;i}x^\gamma_{;m}x^\delta_{;j}r_;{}^m,\end{aligned} \tag{3.5}$$

$$\begin{aligned}\mathcal{L}F &= -\frac{2n}{F^3}F^{ij}F_{;i}F_{;j} - \frac{n}{F}\left(F^{ij}h_{jk}h_i^k - \frac{1}{n}F^2\right) + \frac{u^2\lambda''}{\lambda\lambda'^2}F - \frac{u\lambda''}{\lambda\lambda'}F^{ij}g_{ij}\\ &\quad + \frac{u}{\lambda'^2}\left(\frac{\lambda'\lambda''}{\lambda} - \lambda''' + \frac{2\lambda''^2}{\lambda'}\right)F^{ij}r_{;i}r_{;j} - \frac{2\lambda''}{\lambda'^2}F^{ij}u_{;i}r_{;j}\\ &\quad - \frac{\lambda}{\lambda'}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{;i}x^\gamma_{;m}x^\delta_{;j}r_;{}^m - \left(\frac{n}{F} - \frac{u}{\lambda'}\right)F^{ij}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{;i}\nu^\beta\nu^\gamma x^\delta_{;j}.\end{aligned} \tag{3.6}$$

Proof. The 0-component of (3.1) gives

$$\dot r = \mathcal{F}v^{-1} = \left(\frac{n}{F} - \frac{u}{\lambda'}\right) v^{-1},$$

while from (2.8) we see, using the 1-homogeneity of F,

$$-\frac{n}{F^2}F^{ij}r_{;ij} = \frac{n}{vF} - \frac{n\lambda'}{\lambda F^2}F^{ij}g_{ij} + \frac{n\lambda'}{\lambda F^2}F^{ij}r_{;i}r_{;j}.$$

Adding up gives (3.4).

To prove (3.5), note that $\lambda\partial_r$ is a conformal vector field, i.e. for all ambient vector fields $\bar X$ there holds

$$\bar\nabla_{\bar X}(\lambda\partial_r) = \lambda'\bar X.$$

Hence

$$\dot u = \bar g(\lambda'\dot x, \nu) + \bar g(\lambda\partial_r, \bar\nabla_{\dot x}\nu) = \lambda'\mathcal{F} - \bar g(\lambda\partial_r, \operatorname{grad}\mathcal{F}).$$

Furthermore there holds

$$Xu = \bar g(\lambda\partial_r, A(X))$$

and

$$\begin{aligned}\nabla^2 u(X,Y) &= Y(Xu) - (\nabla_Y X)u\\ &= \lambda' h(X,Y) + \bar g(\lambda\partial_r, \nabla_Y A(X)) - h(Y, A(X))u \quad \forall X,Y\in TM.\end{aligned}$$

We use the Codazzi equation (2.5) to deduce

$$\begin{aligned}\bar g(\lambda\partial_r, \nabla_Y A(X)) &= \lambda\bar g_{\alpha\beta}r^\alpha x^\beta_{\ ;k}h^k_{i;j}X^iY^j\\ &= \lambda\bar g_{\alpha\beta}r^\alpha x^\beta_{\ ;k}h_{ij;}{}^kX^iY^j - \lambda\bar g_{\alpha\beta}r^\alpha x^\beta_{\ ;k}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;i}g^{km}x^\gamma_{\ ;m}x^\delta_{\ ;j}.\end{aligned}$$

Note

$$\bar g_{\alpha\beta}r^\alpha x^\beta_{\ ;k} = r_{;k},$$

we thus get

$$u_{;ij} = \lambda' h_{ij} + \lambda r_{;k}h_{ij;}{}^k - h^k_i h_{kj}u - \lambda\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;i}x^\gamma_{\ ;m}x^\delta_{\ ;j}r_;{}^m. \tag{3.7}$$

Since

$$\mathcal{F}_{;k} = -\frac{n}{F^2}F^{ij}h_{ij;k} - \frac{u_{;k}}{\lambda'} + \frac{\lambda'' u}{\lambda'^2}r_{;k},$$

we obtain (3.5).

From (3.2), we have

$$\begin{aligned}\dot F &= -F^{ij}\mathcal{F}_{;ij} - F^{ij}\mathcal{F}h_{jk}h^k_i - F^{ij}\mathcal{F}\bar R_{\alpha\beta\gamma\delta}x^\alpha_{\ ;i}\nu^\beta\nu^\gamma x^\delta_{\ ;j}\\ &= \frac{n}{F^2}F^{ij}F_{;ij} - \frac{2n}{F^3}F^{ij}F_{;i}F_{;j} + \frac{1}{\lambda'}F^{ij}u_{;ij} - \frac{u}{\lambda'^2}F^{ij}\lambda'_{;ij} + \frac{2u}{\lambda'^3}F^{ij}\lambda'_{;i}\lambda'_{;j}\\ &\quad - \frac{2\lambda''}{\lambda'^2}F^{ij}u_{;i}r_{;j} - \left(\frac{n}{F} - \frac{u}{\lambda'}\right)F^{ij}h_{jk}h^k_i - \left(\frac{n}{F} - \frac{u}{\lambda'}\right)F^{ij}\bar R_{\alpha\beta\gamma\delta}x^\alpha_{\ ;i}\nu^\beta\nu^\gamma x^\delta_{\ ;j}.\end{aligned}$$

Using (3.7) and (2.8), we get (3.6). ☐

We also need the parabolic equation satisfied by the Weingarten operator. A similar calculation was performed in [20, Lemma 2.4.1], but since our flow speed is not directly covered by this reference, we deduce it for convenience.

3.3. **Lemma.** *Along* (3.1) *the following evolution equation holds.*

$$\begin{aligned}
\mathcal{L}h_i^j =& -\frac{2n}{F^3}F_{;i}F_;{}^j + \frac{n}{F^2}F^{kl,rs}h_{kl;i}h_{rs;}{}^j - \frac{\lambda''}{\lambda'^2}\left(u_{;i}r_;{}^j + r_{;i}u_;{}^j\right) \\
&- \frac{u}{\lambda'^2}\left(\lambda''' - \frac{2\lambda''^2}{\lambda'} - \frac{\lambda''\lambda'}{\lambda}\right)r_{;i}r_;{}^j + \left(1 + \frac{u\lambda''}{\lambda'^2 v}\right)h_i^j \\
&- \frac{u\lambda''}{\lambda'\lambda}\delta_i^j - \frac{\lambda}{\lambda'}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta{}_{;i}x^\gamma{}_{;m}x^\delta{}_{;l}r_;{}^m g^{lj} \\
&- \left(\frac{n}{F} - \frac{u}{\lambda'}\right)\bar{R}_{\alpha\beta\gamma\delta}x^\alpha{}_{;i}\nu^\beta\nu^\gamma x^\delta{}_{;m}g^{mj} + \frac{n}{F^2}F^{kl}h_{rk}h_l^r h_i^j - \frac{2n}{F}h_k^j h_i^k \\
&+ \frac{n}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta}\left(x^\alpha{}_{;l}x^\beta{}_{;r}x^\gamma{}_{;k}x^\delta{}_{;m}h_i^m + x^\alpha{}_{;l}x^\beta{}_{;i}x^\gamma{}_{;k}x^\delta{}_{;m}h_r^m\right)g^{rj} \\
&+ \frac{2n}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha{}_{;l}x^\beta{}_{;r}x^\gamma{}_{;i}x^\delta{}_{;m}h_k^m g^{rj} + \frac{n}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta{}_{;k}x^\gamma{}_{;l}\nu^\delta h_i^j \\
&+ \frac{n}{F}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta{}_{;i}\nu^\gamma x^\delta{}_{;m}g^{mj} - \frac{n}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta{}_{;k}x^\gamma{}_{;l}x^\delta{}_{;i}x^\epsilon{}_{;m}g^{mj} \\
&- \frac{n}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta{}_{;i}x^\gamma{}_{;k}x^\delta{}_{;m}x^\epsilon{}_{;l}g^{mj}.
\end{aligned} \tag{3.8}$$

Proof. We use (3.3) and calculate $-\mathcal{F}_{;ij}$ step by step. We use

$$\mathcal{F} = \frac{n}{F} - \frac{u}{\lambda'}, \quad -\mathcal{F}_{;i} = \frac{n}{F^2}F_{;i} + \frac{u_{;i}}{\lambda'} - \frac{u\lambda'' r_{;i}}{\lambda'^2},$$

(3.7) as well as (2.8), to deduce

$$\begin{aligned}
-\mathcal{F}_{;ij} =& -\frac{2n}{F^3}F_{;i}F_{;j} + \frac{n}{F^2}F_{;ij} + \frac{u_{;ij}}{\lambda'} - \frac{\lambda''}{\lambda'^2}\left(u_{;i}r_{;j} + r_{;i}u_{;j}\right) \\
&- u\left(\frac{\lambda'''}{\lambda'^2} - \frac{2\lambda''^2}{\lambda'^3}\right)r_{;i}r_{;j} - \frac{u\lambda''}{\lambda'^2}r_{;ij} \\
=& -\frac{2n}{F^3}F_{;i}F_{;j} + \frac{n}{F^2}F_{;ij} - \frac{\lambda''}{\lambda'^2}\left(u_{;i}r_{;j} + r_{;i}u_{;j}\right) \\
&- u\left(\frac{\lambda'''}{\lambda'^2} - \frac{2\lambda''^2}{\lambda'^3}\right)r_{;i}r_{;j} + h_{ij} + \frac{\lambda}{\lambda'}r_{;k}h_{ij;}{}^k - \frac{u}{\lambda'}h_{ik}h_j^k \\
&- \frac{\lambda}{\lambda'}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta{}_{;i}x^\gamma{}_{;m}x^\delta{}_{;j}r_;{}^m + \frac{u\lambda''}{\lambda'^2}\left(v^{-1}h_{ij} - \frac{\lambda'}{\lambda}g_{ij} + \frac{\lambda'}{\lambda}r_{;i}r_{;j}\right) \\
=& -\frac{2n}{F^3}F_{;i}F_{;j} + \frac{n}{F^2}F_{;ij} - \frac{\lambda''}{\lambda'^2}\left(u_{;i}r_{;j} + r_{;i}u_{;j}\right) + \frac{\lambda}{\lambda'}r_{;k}h_{ij;}{}^k \\
&- \frac{u}{\lambda'^2}\left(\lambda''' - \frac{2\lambda''^2}{\lambda'} - \frac{\lambda''\lambda'}{\lambda}\right)r_{;i}r_{;j} + \left(1 + \frac{u\lambda''}{\lambda'^2 v}\right)h_{ij} - \frac{u}{\lambda'}h_{ik}h_j^k \\
&- \frac{u\lambda''}{\lambda'\lambda}g_{ij} - \frac{\lambda}{\lambda'}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta{}_{;i}x^\gamma{}_{;m}x^\delta{}_{;j}r_;{}^m.
\end{aligned} \tag{3.9}$$

We have to transform $F_{;ij}$. Using the Codazzi equation (2.5)and the Ricci identities (2.2), we obtain

$$\begin{aligned}
F_{;ij} &= F^{kl,rs}h_{kl;i}h_{rs;j} + F^{kl}h_{kl;ij}\\
&= F^{kl,rs}h_{kl;i}h_{rs;j} + F^{kl}h_{ki;lj} - F^{kl}\left(\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;k}x^\gamma_{\ ;l}x^\delta_{\ ;i}\right)_{;j}\\
&= F^{kl,rs}h_{kl;i}h_{rs;j} + F^{kl}h_{ki;jl} + F^{kl}R_{ljk}{}^a h_{ai} + F^{kl}R_{lji}{}^a h_{ka}\\
&\quad - F^{kl}\left(\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;k}x^\gamma_{\ ;l}x^\delta_{\ ;i}\right)_{;j}\\
&= F^{kl,rs}h_{kl;i}h_{rs;j} + F^{kl}R_{ljk}{}^a h_{ai} + F^{kl}R_{lji}{}^a h_{ka} + F^{kl}h_{ij;kl}\\
&\quad - F^{kl}\left(\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;k}x^\gamma_{\ ;l}x^\delta_{\ ;i}\right)_{;j} - F^{kl}\left(\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;i}x^\gamma_{\ ;k}x^\delta_{\ ;j}\right)_{;l}
\end{aligned}$$

Differentiating the big brackets by the product rule gives, using the Weingarten equation (2.4) and the Gauss equation (2.6)

$$\begin{aligned}
F_{;ij} &= F^{kl}h_{ij;kl} + F^{kl,rs}h_{kl;i}h_{rs;j}\\
&\quad + F^{kl}(h_{la}h_{jk} - h_{lk}h_{ja} + \bar R_{\alpha\beta\gamma\delta}x^\alpha_{\ ;l}x^\beta_{\ ;j}x^\gamma_{\ ;k}x^\delta_{\ ;a})h^a_i\\
&\quad + F^{kl}(h_{la}h_{ji} - h_{li}h_{ja} + \bar R_{\alpha\beta\gamma\delta}x^\alpha_{\ ;l}x^\beta_{\ ;j}x^\gamma_{\ ;i}x^\delta_{\ ;a})h^a_k\\
&\quad - F^{kl}\bar R_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta_{\ ;k}x^\gamma_{\ ;l}x^\delta_{\ ;i}x^\epsilon_{\ ;j} - F^{kl}\bar R_{\alpha\beta\gamma\delta}x^\alpha_{\ ;m}x^\beta_{\ ;k}x^\gamma_{\ ;l}x^\delta_{\ ;i}h^m_j\\
&\quad + F^{kl}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;k}\nu^\gamma x^\delta_{\ ;i}h_{lj} + F^{kl}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;k}x^\gamma_{\ ;l}\nu^\delta h_{ij}\\
&\quad - F^{kl}\bar R_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta_{\ ;i}x^\gamma_{\ ;k}x^\delta_{\ ;j}x^\epsilon_{\ ;l} - F^{kl}\bar R_{\alpha\beta\gamma\delta}x^\alpha_{\ ;m}x^\beta_{\ ;i}x^\gamma_{\ ;k}x^\delta_{\ ;j}h^m_l\\
&\quad + F^{kl}h_{kl}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;i}\nu^\gamma x^\delta_{\ ;j} + F^{kl}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;i}x^\gamma_{\ ;k}\nu^\delta h_{jl}
\end{aligned}$$

and after some rearranging, using the homogeneity of F,

$$\begin{aligned}
F_{;ij} &= F^{kl}h_{ij;kl} + F^{kl,rs}h_{kl;i}h_{rs;j} + F^{kl}h_{rk}h^r_l h_{ij} - Fh_{ik}h^k_j\\
&\quad + F^{kl}\bar R_{\alpha\beta\gamma\delta}\left(x^\alpha_{\ ;l}x^\beta_{\ ;j}x^\gamma_{\ ;k}x^\delta_{\ ;m}h^m_i + x^\alpha_{\ ;l}x^\beta_{\ ;i}x^\gamma_{\ ;k}x^\delta_{\ ;m}h^m_j\right)\\
&\quad + 2F^{kl}\bar R_{\alpha\beta\gamma\delta}x^\alpha_{\ ;l}x^\beta_{\ ;j}x^\gamma_{\ ;i}x^\delta_{\ ;m}h^m_k + F^{kl}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;k}x^\gamma_{\ ;l}\nu^\delta h_{ij}\\
&\quad + F\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;i}\nu^\gamma x^\delta_{\ ;j} - F^{kl}\bar R_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta_{\ ;k}x^\gamma_{\ ;l}x^\delta_{\ ;i}x^\epsilon_{\ ;j}\\
&\quad - F^{kl}\bar R_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta_{\ ;i}x^\gamma_{\ ;k}x^\delta_{\ ;j}x^\epsilon_{\ ;l}.
\end{aligned} \tag{3.10}$$

From (3.3), inserting (3.9), we get

$$\begin{aligned}
\dot h_{ij} &= -\mathcal F_{;ij} + \mathcal F h_{ik}h^k_j - \mathcal F\bar R_{\alpha\beta\gamma\delta}x^\alpha_{\ ;i}\nu^\beta\nu^\gamma x^\delta_{\ ;j}\\
&= -\frac{2n}{F^3}F_{;i}F_{;j} + \frac{n}{F^2}F_{;ij} - \frac{\lambda''}{\lambda'^2}\left(u_{;i}r_{;j} + r_{;i}u_{;j}\right) + \frac{\lambda}{\lambda'}r_{;k}h_{ij;}{}^k\\
&\quad - \frac{u}{\lambda'^2}\left(\lambda''' - \frac{2\lambda''^2}{\lambda'} - \frac{\lambda''\lambda'}{\lambda}\right)r_{;i}r_{;j} + \left(1 + \frac{u\lambda''}{\lambda'^2 v}\right)h_{ij} - \frac{u}{\lambda'}h_{ik}h^k_j - \frac{u\lambda''}{\lambda'\lambda}g_{ij}\\
&\quad - \frac{\lambda}{\lambda'}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;i}x^\gamma_{\ ;m}x^\delta_{\ ;j}r_;{}^m + \left(\frac{n}{F} - \frac{u}{\lambda'}\right)h_{ik}h^k_j\\
&\quad - \left(\frac{n}{F} - \frac{u}{\lambda'}\right)\bar R_{\alpha\beta\gamma\delta}x^\alpha_{\ ;i}\nu^\beta\nu^\gamma x^\delta_{\ ;j}.
\end{aligned}$$

Inserting (3.10) into this equation gives

$$\begin{aligned}
\dot{h}_{ij} &= \frac{n}{F^2}F^{kl}h_{ij;kl} + \frac{\lambda}{\lambda'}r_{;k}h_{ij;}{}^{k} - \frac{2n}{F^3}F_{;i}F_{;j} + \frac{n}{F^2}F^{kl,rs}h_{kl;i}h_{rs;j} \\
&\quad - \frac{\lambda''}{\lambda'^2}\left(u_{;i}r_{;j} + r_{;i}u_{;j}\right) - \frac{u}{\lambda'^2}\left(\lambda''' - \frac{2\lambda''^2}{\lambda'} - \frac{\lambda''\lambda'}{\lambda}\right)r_{;i}r_{;j} + \left(1 + \frac{u\lambda''}{\lambda'^2 v}\right)h_{ij} \\
&\quad - \frac{2u}{\lambda'}h_{ik}h^k_j - \frac{u\lambda''}{\lambda'\lambda}g_{ij} - \frac{\lambda}{\lambda'}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;i}x^\gamma_{\ ;m}x^\delta_{\ ;j}r_;{}^m \\
&\quad - \left(\frac{n}{F} - \frac{u}{\lambda'}\right)\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{\ ;i}\nu^\beta\nu^\gamma x^\delta_{\ ;j} + \frac{n}{F^2}F^{kl}h_{rk}h^r_l h_{ij} \\
&\quad + \frac{n}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta}\left(x^\alpha_{\ ;l}x^\beta_{\ ;j}x^\gamma_{\ ;k}x^\delta_{\ ;m}h^m_i + x^\alpha_{\ ;l}x^\beta_{\ ;i}x^\gamma_{\ ;k}x^\delta_{\ ;m}h^m_j\right) \\
&\quad + \frac{2n}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{\ ;l}x^\beta_{\ ;j}x^\gamma_{\ ;i}x^\delta_{\ ;m}h^m_k + \frac{n}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;k}x^\gamma_{\ ;l}\nu^\delta h_{ij} \\
&\quad + \frac{n}{F}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;i}\nu^\gamma x^\delta_{\ ;j} - \frac{n}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta_{\ ;k}x^\gamma_{\ ;l}x^\delta_{\ ;i}x^\epsilon_{\ ;j} \\
&\quad - \frac{n}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x^\beta_{\ ;i}x^\gamma_{\ ;k}x^\delta_{\ ;j}x^\epsilon_{\ ;l}.
\end{aligned}$$

Using

$$\dot{h}^i_j = \dot{g}^{ik}h_{kj} + g^{ik}\dot{h}_{kj} = -g^{il}\dot{g}_{lm}g^{mk}h_{kj} + g^{ik}\dot{h}_{kj} = 2\left(\frac{u}{\lambda'} - \frac{n}{F}\right)h^i_k h^k_j + g^{ik}\dot{h}_{kj}$$

gives the result. □

In particular, when the ambient space N is a space form of sectional curvature K_N, then

$$\bar{R}_{\alpha\beta\gamma\delta} = K_N(\bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma} - \bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta})$$

and

$$\lambda'' = -K_N\lambda, \quad \lambda''' = -K_N\lambda' = \frac{\lambda''\lambda'}{\lambda}$$

and (3.8) reduces to

$$\begin{aligned}
\mathcal{L}h^j_i &= -\frac{2n}{F^3}F_{;i}F_;{}^j + \frac{n}{F^2}F^{kl,rs}h_{kl;i}h_{rs;}{}^j - \frac{\lambda''}{\lambda'^2}\left(u_{;i}r_;{}^j + r_{;i}u_;{}^j\right) \\
&\quad - \frac{u}{\lambda'^2}\left(\lambda''' - \frac{2\lambda''^2}{\lambda'} - \frac{\lambda''\lambda'}{\lambda}\right)r_{;i}r_;{}^j + \left(1 + \frac{u\lambda''}{\lambda'^2 v}\right)h^j_i \\
&\quad - \frac{u\lambda''}{\lambda'\lambda}\delta^j_i - \left(\frac{n}{F} - \frac{u}{\lambda'}\right)K_N\delta^j_i + \frac{n}{F^2}F^{kl}h_{rk}h^r_l h^j_i - \frac{2n}{F}h^j_k h^k_i \\
&\quad + \frac{n}{F^2}F^{kl}K_N(h_{il}\delta^j_k + h^j_l g_{ik} - 2g_{kl}h^j_i) \\
&\quad + \frac{2n}{F^2}F^{kl}K_N(h_{kl}\delta^j_i - g_{li}h^j_k) + K_N\frac{n}{F^2}F^{kl}g_{kl}h^j_i - K_N\frac{n}{F}\delta^j_i \\
&= -\frac{2n}{F^3}F_{;i}F_;{}^j + \frac{n}{F^2}F^{kl,rs}h_{kl;i}h_{rs;}{}^j + K_N\frac{\lambda}{\lambda'^2}\left(u_{;i}r_;{}^j + r_{;i}u_;{}^j\right) \\
&\quad + K_N^2\frac{2u\lambda^2}{\lambda'^3}r_{;i}r_;{}^j + \left(1 - K_N\frac{u^2}{\lambda'^2} + \frac{n}{F^2}F^{kl}h_{rk}h^r_l - \frac{n}{F^2}K_N F^{kl}g_{kl}\right)h^j_i \\
&\quad + 2\frac{u}{\lambda'}K_N\delta^j_i - \frac{2n}{F}h^j_k h^k_i.
\end{aligned} \tag{3.11}$$

4. Upper bounds for the curvature function

In this section we show that the curvature function F is bounded from above along the flow (3.1) in the case $F = n\frac{H_k}{H_{k-1}}$ for very general λ. For this paper, we only apply it in the case $\lambda = \sin$, but due to its generality it might be of use in further situations. For the rest of the paper, whenever we stipulate the existence of a generic constant, it will be allowed to depend on the data of the problem, i.e. on N, M_0 and F, unless otherwise specified.

4.1. **Proposition.** *Let $a, b \in \mathbb{R}$ and $(N, \bar{g})$ be the warped space $((a,b)\times\mathbb{S}^n, dr^2 + \lambda^2(r)\sigma)$ with $\lambda, \lambda' > 0$. Let*

$$F = n\frac{H_k}{H_{k-1}}$$

and let $x_0(M)$be the embedding of a closed n-dimensional manifold M into N, such that $x_0(M)$ is a graph over the domain $\mathbb{S}^n$ and such that $\kappa \in \Gamma_k$ for all n-tuples of principal curvatures along $x_0(M)$. Then along any solution x of (3.1) with initial embedding x_0 there exists a constant c, such that

$$F \leq c.$$

4.2. *Remark.* Note that under the hypothesis of Proposition 4.1 we have

$$r \leq \sup r_0, \quad r \geq \inf r_0$$

along the flow, due to the maximum principle. This assertion also holds for arbitrary monotone curvature functions F.

Now we prove Proposition 4.1.

Proof. Consider the test function

$$\Phi = \log F + \frac{u}{\lambda} + \alpha r$$

with a large constant α to be determined. Assume Φ attains its maximum at p. By a suitable choice of coordinate we can assume $g_{ij}|_p = \delta_{ij}$, $h_{ij}|_p$ is diagonal and in turn F^{ij} is diagonal at p. Assume $F|_p \geq C$ for some sufficient large constant C. In the following, we compute at p.

From Lemma 3.2 we deduce

$$\begin{aligned}
\mathcal{L}\left(\frac{u}{\lambda}\right) &= \frac{1}{\lambda}\mathcal{L}u - \frac{u\lambda'}{\lambda^2}\mathcal{L}r + \frac{n}{F^2}\frac{2\lambda'}{\lambda^2}F^{ij}u_{;i}r_{;j} - \frac{n}{F^2}\frac{u}{\lambda^3}\left(2\lambda'^2 - \lambda\lambda''\right)F^{ij}r_{;i}r_{;j} \\
&= \frac{n}{F^2}\left(F^{ij}h_{ik}h^k_j - \frac{1}{n}F^2\right)\bar{g}(\partial_r, \nu) - \frac{\lambda''}{\lambda'^2}\|\nabla r\|^2 u \\
&\quad - \frac{u\lambda'}{\lambda^2}\left(\frac{2n}{vF} - \frac{\lambda}{\lambda'} - \frac{n\lambda'}{\lambda F^2}F^{ij}g_{ij} + \frac{n\lambda'}{\lambda F^2}F^{ij}r_{;i}r_{;j}\right) \\
&\quad + \frac{n}{F^2}\frac{2\lambda'}{\lambda^2}F^{ij}u_{;i}r_{;j} - \frac{n}{F^2}\frac{u}{\lambda^3}\left(2\lambda'^2 - \lambda\lambda''\right)F^{ij}r_{;i}r_{;j} \\
&\quad + \frac{n}{F^2}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;i}x^\gamma_{\ ;m}x^\delta_{\ ;j}r_;^{\ m} \\
&\leq \frac{n}{F^2}\left(F^{ii}h^2_{ii} - \frac{1}{n}F^2\right)\bar{g}(\partial_r, \nu) + \frac{C}{F}F^{ii}|u_{;i}||r_{;i}| + C + \sum_i CF^{ii}.
\end{aligned}$$

We also have

$$\begin{aligned}
\mathcal{L}\log F &= -\frac{n}{F^2}F^{ij}(\log F)_{;i}(\log F)_{;j} - \frac{n}{F^2}\left(F^{ij}h_{jk}h_i^k - \frac{1}{n}F^2\right) + \frac{u^2\lambda''}{\lambda\lambda'^2} \\
&\quad - \frac{u\lambda''}{\lambda\lambda' F}F^{ij}g_{ij} + \frac{u}{\lambda'^2 F}\left(\frac{\lambda'\lambda''}{\lambda} - \lambda''' + \frac{2\lambda''^2}{\lambda'}\right)F^{ij}r_{;i}r_{;j} - \frac{2\lambda''}{\lambda'^2 F}F^{ij}u_{;i}r_{;j} \\
&\quad - \frac{\lambda}{\lambda' F}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_{\ ;i}x^\gamma_{\ ;m}x^\delta_{\ ;j}r_;{}^m - \frac{1}{F}\left(\frac{n}{F} - \frac{u}{\lambda'}\right)F^{ij}\bar{R}_{\alpha\beta\gamma\delta}x^\alpha_{\ ;i}\nu^\beta\nu^\gamma x^\delta_{\ ;j} \\
&\le -\frac{n}{F^2}F^{ii}(\log F)_{;i}(\log F)_{;i} - \frac{n}{F^2}\left(F^{ii}h_{ii}^2 - \frac{1}{n}F^2\right) \\
&\quad + \frac{C}{F}F^{ii}|u_{;i}||r_{;i}| + C + \sum_i CF^{ii}.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathcal{L}\Phi &= \mathcal{L}\log F + \mathcal{L}\left(\frac{u}{\lambda}\right) + \alpha\mathcal{L}r \\
&\le -\frac{n}{F^2}F^{ii}(\log F)_{;i}(\log F)_{;i} - \frac{n}{F^2}\left(F^{ii}h_{ii}^2 - \frac{1}{n}F^2\right)(1 - \bar{g}(\partial_r, \nu)) \\
&\quad - \alpha\frac{\lambda}{\lambda'} + \alpha\frac{C}{F} + \alpha\frac{C}{F}\sum_i F^{ii} + \frac{C}{F}F^{ii}|u_{;i}||r_{;i}| + C + \sum_i CF^{ii}.
\end{aligned}$$

A calculation using the Newton-MacLaurin inequalities gives for our special F:

$$F^{ii}h_{ii}^2 - \frac{1}{n}F^2 \ge 0$$

and

$$F_i^i \le C(n,k),$$

see [33, Prop. 2.2] for useful formulas for this calculation.

Thus

$$\mathcal{L}\Phi \le -\frac{n}{F^2}F^{ii}(\log F)_{;i}(\log F)_{;i} - \alpha\frac{\lambda}{\lambda'} + \alpha\frac{C}{F} + \frac{C}{F}F^{ii}|u_{;i}||r_{;i}| + C.$$

From the maximal property of Φ at p, we have

$$\nabla \log F = -\frac{1}{\lambda}\nabla u + \frac{u\lambda'}{\lambda^2}\nabla r - \alpha\nabla r.$$

Therefore

$$\begin{aligned}
0 \le \mathcal{L}\Phi &\le -\frac{n}{F^2}F^{ii}\left(-\frac{1}{\lambda}u_{;i} + \frac{u\lambda'}{\lambda^2}r_{;i} - \alpha r_{;i}\right)^2 - \alpha\frac{\lambda}{\lambda'} + \alpha\frac{C}{F} + \frac{C}{F}F^{ii}|u_{;i}||r_{;i}| + C \\
&\le -\frac{n}{2F^2\lambda^2}F^{ii}{u_{;i}}^2 + \frac{n}{F^2}F^{ii}\left(\frac{u\lambda'}{\lambda^2}r_{;i} - \alpha r_{;i}\right)^2 + \frac{C}{F}F^{ii}|u_{;i}||r_{;i}| \\
&\quad - \alpha\frac{\lambda}{\lambda'} + \alpha\frac{C}{F} + C \\
&\le -\frac{n}{2F^2\lambda^2}F^{ii}\left(|u_{;i}| - \frac{CF\lambda^2}{n}|r_{;i}|\right)^2 - \alpha\frac{\lambda}{\lambda'} + \alpha\frac{C}{F} + C + C\frac{\alpha^2}{F^2} \\
&\le -\alpha\frac{\lambda}{\lambda'} + \alpha\frac{C}{F} + C + C\frac{\alpha^2}{F^2}.
\end{aligned}$$

Assume $F|_p \geq \alpha$. Then by choosing α large enough, we get the RHS of above inequality is negative, a contradiction. Therefore, $F|_p \leq \alpha$ for our choice of α and in turn $\Phi|_p$ is bounded. Since Φ attains its maximum at p, we conclude that F is bounded from above. □

5. Gradient estimates

In this section we show that the graph function has a uniform C^1 bound along the flow (3.1) for very general F and λ.

5.1. **Proposition.** *Let $a, b \in \mathbb{R}$ and $(N, \bar{g})$ be the warped space $((a,b)\times \mathbb{S}^n, dr^2 + \lambda^2(r)\sigma)$ with $\lambda, \lambda' > 0$. Let $F \in C^\infty(\Gamma)$ be a positive, 1-homogeneous and strictly monotone curvature function and let $x_0(M)$ be the embedding of a closed n-dimensional manifold M into N, such that $x_0(M)$ is a graph over the domain $\mathbb{S}^n$ with graph function r and such that $\kappa \in \Gamma$ for all n-tuples of principal curvatures along $x_0(M)$. Then along any solution x of (3.1) with initial embedding x_0, there exists a constant c, such that*

$$|\hat{\nabla} r| \leq c.$$

Proof. Recall

$$\varphi = \int_a^r \frac{1}{\lambda(s)}\, ds \tag{5.1}$$

To simplify the notation, we just use $\varphi_i = \varphi_{,i}$, etc., i.e., we omit the comma when taking covariant derivative on $\mathbb{S}^n$.

We rewrite the flow equation as a scalar equation for φ:

$$\begin{aligned}
\partial_t \varphi &= \frac{1}{\lambda}\left(\frac{n}{F\left(\frac{\lambda'}{\lambda v}\delta^j_i - \frac{1}{\lambda v}\tilde{g}^{jk}\varphi_{ki}\right)} - \frac{u}{\lambda'}\right) v \\
&= \frac{nv^2}{F(\lambda'\delta^j_i - \tilde{g}^{jk}\varphi_{ki})} - \frac{1}{\lambda'} =: G(\varphi, \hat{\nabla}\varphi, \hat{\nabla}^2\varphi),
\end{aligned} \tag{5.2}$$

where $\tilde{g}^{ij} = \sigma^{ij} - \frac{\varphi^i\varphi^j}{v^2}$. For simplicity, we denote by $F = F(\lambda'\delta^j_i - \tilde{g}^{jk}\varphi_{ki})$ and F^i_j the derivative of F with respect to its argument.

We compute

$$\begin{aligned}
G^{ij} &:= \frac{\partial G}{\partial \varphi_{ij}} = \frac{nv^2}{F^2} F^i_k \tilde{g}^{kj}, \\
G^{\varphi_p} &:= \frac{\partial G}{\partial \varphi_p} = \frac{2n\varphi^p}{F} + \frac{nv^2}{F^2} F^i_j \left(-\frac{\sigma^{jp}\varphi^k + \sigma^{kp}\varphi^j}{v^2} + \frac{2\varphi^j\varphi^k\varphi^p}{v^4}\right)\varphi_{ki}, \\
G^{\varphi} &:= \frac{\partial G}{\partial \varphi} = -\frac{nv^2\lambda''\lambda}{F^2} F^i_i + \frac{\lambda''\lambda}{\lambda'^2}.
\end{aligned} \tag{5.3}$$

Using the 1-homogeneity of F, we have

$$\begin{aligned}
G^{ij}\varphi_{ij} &= \frac{nv^2}{F^2} F^i_k \tilde{g}^{kj}\varphi_{ij} \\
&= -\frac{nv^2}{F^2} F^i_k(\lambda'\delta^k_i - \tilde{g}^{kj}\varphi_{ij}) + \frac{nv^2\lambda'}{F^2} F^i_i = -\frac{nv^2}{F} + \frac{nv^2\lambda'}{F^2} F^i_i
\end{aligned} \tag{5.4}$$

and

$$G^{ij}\varphi_i\varphi_j = \frac{nv^2}{F^2} F^i_k \tilde{g}^{kj}\varphi_i\varphi_j = \frac{n}{F^2} F^i_k \varphi^k \varphi_i. \tag{5.5}$$

Let $\mathcal{L} = \partial_t - G^{ij}\nabla^2_{ij}$ be the parabolic operator. Using the Ricci identities on $\mathbb{S}^n$ we get

$$\begin{aligned}\mathcal{L}|\hat\nabla\varphi|^2 = & -2G^{ij}\varphi_{ik}\varphi_j{}^k - 2G^{ij}\sigma_{ij}|\hat\nabla\varphi|^2 + 2G^{ij}\varphi_i\varphi_j \\ & + G^{\varphi_p}(|\hat\nabla\varphi|^2)_p + 2G^{\varphi}|\hat\nabla\varphi|^2.\end{aligned} \tag{5.6}$$

Let $f\colon [0,\infty) \to (0,\infty)$ be an auxiliary function to be determined. Consider a test function

$$\Phi = \log\frac{|\hat\nabla\varphi|^2}{f(\varphi)}.$$

In the following we compute at a maximal point of Φ. Due to the maximal property,

$$\hat\nabla|\hat\nabla\varphi|^2 = \frac{f'}{f}|\hat\nabla\varphi|^2\hat\nabla\varphi.$$

By a suitable choice of the coordinates, we may assume $\sigma_{ij} = \delta_{ij}$ and $|\hat\nabla\varphi| = \varphi_1$. Then

$$\varphi_{11} = \frac{1}{2}\frac{f'}{f}|\hat\nabla\varphi|^2, \quad \varphi_{1j} = 0 \text{ for } j = 2, \cdots, n.$$

Then $\tilde g^{ij}$ is diagonal with

$$\tilde g^{11} = \frac{1}{v^2}, \quad \tilde g^{ii} = 1 \text{ for } i \neq 1.$$

We may further assume φ_{ij} is diagonal and in turn F^k_i is diagonal. Thus we have

$$\begin{aligned}-2G^{ij}\varphi_{ik}\varphi_j{}^k &= -\frac{2nv^2}{F^2}F^i_l\tilde g^{lj}\varphi_{ik}\varphi_j{}^k \\ &= -\frac{2n}{F^2}F^{11}\frac{1}{4}\left(\frac{f'}{f}\right)^2|\hat\nabla\varphi|^4 - \frac{2nv^2}{F^2}\sum_{k\geq 2}F^{kk}\varphi^2_{kk},\end{aligned}$$

$$\begin{aligned}-2G^{ij}\sigma_{ij}|\nabla\varphi|^2 + 2G^{ij}\varphi_i\varphi_j &= -\frac{2nv^2}{F^2}F^i_k\tilde g^{kj}(\sigma_{ij}|\hat\nabla\varphi|^2 - \varphi_i\varphi_j) \\ &= -\frac{2nv^2}{F^2}\sum_{k\geq 2}F^{kk}|\hat\nabla\varphi|^2,\end{aligned}$$

$$\begin{aligned}G^{\varphi_p}(|\hat\nabla\varphi|^2)_p &= \left(\frac{2n\varphi^p}{F} + \frac{nv^2}{F^2}F^i_j\left(-\frac{\sigma^{jp}\varphi^k + \sigma^{kp}\varphi^j}{v^2} + \frac{2\varphi^j\varphi^k\varphi^p}{v^4}\right)\varphi_{ki}\right)\frac{f'}{f}|\hat\nabla\varphi|^2\varphi_p \\ &= \frac{f'}{f}\frac{2n}{F}|\hat\nabla\varphi|^4 - \left(\frac{f'}{f}\right)^2\frac{n}{v^2F^2}F^{11}|\hat\nabla\varphi|^6\end{aligned}$$

and

$$2G^{\varphi}|\hat\nabla\varphi|^2 = \left(-\frac{2nv^2\lambda''\lambda}{F^2}F^i_i + \frac{2\lambda''\lambda}{\lambda'^2}\right)|\hat\nabla\varphi|^2.$$

On the other hand, using (5.2), (5.4) and (5.5) we get

$$\begin{aligned}\mathcal{L}(f(\varphi)) &= f'\left(\frac{nv^2}{F} - \frac{1}{\lambda'}\right) - f'G^{ij}\varphi_{ij} - f''G^{ij}\varphi_i\varphi_j \\ &= f'\left(\frac{2nv^2}{F} - \frac{1}{\lambda'}\right) - f'\frac{nv^2\lambda'}{F^2}F^i_i - f''\frac{n}{F^2}F^{11}|\hat\nabla\varphi|^2.\end{aligned} \tag{5.7}$$

Using (5.6)–(5.7) and the maximal property of Φ at p, we have

$$
\begin{aligned}
0 \le{}& \frac{\mathcal{L}(|\hat\nabla\varphi|^2)}{|\nabla\varphi|^2} - \frac{\mathcal{L}f}{f}\\
={}& -\frac{2n}{F^2}F^{11}\frac{1}{4}\left(\frac{f'}{f}\right)^2|\hat\nabla\varphi|^2 - \frac{2nv^2}{F^2|\hat\nabla\varphi|^2}\sum_{k\ge2}F^{kk}\varphi_{kk}^2 - \frac{2nv^2}{F^2}\sum_{k\ge2}F^{kk}\\
&+\frac{f'}{f}\frac{2n}{F}|\hat\nabla\varphi|^2 - \left(\frac{f'}{f}\right)^2\frac{n}{v^2F^2}F^{11}|\hat\nabla\varphi|^4 - \frac{2nv^2\lambda''\lambda}{F^2}F_i^i + 2\frac{\lambda''\lambda}{\lambda'^2}\\
&-\frac{f'}{f}\left(\frac{2nv^2}{F}-\frac{1}{\lambda'}\right) + \frac{f'}{f}\frac{nv^2\lambda'}{F^2}F_i^i + \frac{f''}{f}\frac{n}{F^2}F^{11}|\hat\nabla\varphi|^2\\
={}& -\frac{n}{2F^2}F^{11}\left(\frac{f'}{f}\right)^2|\hat\nabla\varphi|^2 - \frac{2nv^2}{F^2|\hat\nabla\varphi|^2}\sum_{k\ge2}F^{kk}\varphi_{kk}^2 - \frac{2nv^2}{F^2}\sum_{k\ge2}F^{kk} - \frac{f'}{f}\frac{2n}{F}\\
&-\left[\left(\frac{f'}{f}\right)^2\frac{|\hat\nabla\varphi|^2}{v^2} - \frac{f''}{f}\right]\frac{n}{F^2}F^{11}|\hat\nabla\varphi|^2 - \frac{nv^2}{F^2}\left(2\lambda''\lambda - \lambda'\frac{f'}{f}\right)F_i^i + \frac{f'}{f}\frac{1}{\lambda'}\\
&+2\frac{\lambda''\lambda}{\lambda'^2}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
-\frac{f'}{f}\frac{2n}{F} &= -\frac{f'}{f}\frac{2n}{F^2}F_k^i(\lambda'\delta_i^k - \tilde g^{kj}\varphi_{ij})\\
&= -\frac{f'}{f}\frac{2n\lambda'}{F^2}F_i^i + \left(\frac{f'}{f}\right)^2\frac{n}{F^2}\frac{1}{v^2}F^{11}|\hat\nabla\varphi|^2 + \frac{f'}{f}\frac{2n}{F^2}\sum_{k\ge2}F^{kk}\varphi_{kk}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
0 \le{}& -\frac{2n}{F^2}F^{11}\frac{1}{4}\left(\frac{f'}{f}\right)^2|\hat\nabla\varphi|^2 - \frac{2nv^2}{F^2|\hat\nabla\varphi|^2}\sum_{k\ge2}F^{kk}\varphi_{kk}^2 - \frac{2nv^2}{F^2}\sum_{k\ge2}F^{kk}\\
&-\frac{f'}{f}\frac{2n\lambda'}{F^2}F_i^i + \left(\frac{f'}{f}\right)^2\frac{n}{F^2}\frac{1}{v^2}F^{11}|\hat\nabla\varphi|^2 + \frac{f'}{f}\frac{2n}{F^2}\sum_{k\ge2}F^{kk}\varphi_{kk} + 2\frac{\lambda''\lambda}{\lambda'^2}\\
&-\left[\left(\frac{f'}{f}\right)^2\frac{|\hat\nabla\varphi|^2}{v^2} - \frac{f''}{f}\right]\frac{n}{F^2}F^{11}|\hat\nabla\varphi|^2 - \frac{nv^2}{F^2}\left(2\lambda''\lambda - \lambda'\frac{f'}{f}\right)F_i^i + \frac{f'}{f}\frac{1}{\lambda'}
\end{aligned}
$$

and, completing the square,

$$
\begin{aligned}
0 \le{}& -\frac{n}{F^2}F^{11}\left[\frac{1}{2}\left(\frac{f'}{f}\right)^2 + \left(\frac{f'}{f}\right)^2\frac{|\hat\nabla\varphi|^2}{v^2} - \frac{f''}{f} + 2\lambda''\lambda - \lambda'\frac{f'}{f}\right]|\hat\nabla\varphi|^2\\
&+\frac{n}{F^2}F^{11}\left[-2\lambda'\frac{f'}{f} + \left(\frac{f'}{f}\right)^2\frac{|\hat\nabla\varphi|^2}{v^2} - 2\lambda''\lambda + \lambda'\frac{f'}{f}\right]\\
&-\frac{2n}{F^2}\sum_{k\ge2}F^{kk}\left(\varphi_{kk} - \frac{1}{2}\frac{f'}{f}\right)^2 - \frac{2n}{F^2|\hat\nabla\varphi|^2}\sum_{k\ge2}F^{kk}\varphi_{kk}^2 + 2\frac{\lambda''\lambda}{\lambda'^2}\\
&+\frac{2n}{F^2}\left(\frac{1}{4}\left(\frac{f'}{f}\right)^2 - \lambda'\frac{f'}{f} - v^2\left(1 + \lambda''\lambda - \frac{1}{2}\lambda'\frac{f'}{f}\right)\right)\sum_{k\ge2}F^{kk} + \frac{f'}{f}\frac{1}{\lambda'}.
\end{aligned}
\tag{5.8}
$$

Choose $f(\varphi) = e^{-a\varphi}$ with $a > 0$ large enough so that the first line on the RHS of (5.8) has a negative sign and when $|\hat{\nabla}\varphi|^2$ is large enough, this line dominates the second line. Also, by choosing $a > 0$ large enough and then $|\hat{\nabla}\varphi|^2$ is large enough, the remaining terms sum up to something negative. We get a contradiction. Thus $|\hat{\nabla}\varphi|^2 \leq C$. $\square$

6. Preserved convexity in the sphere

In ambient spaces where λ'' can be negative it is very difficult to control F from below, if the flow hypersurfaces are not convex. Hence we assume strict convexity in these cases and we have to restrict to space forms to show that this property is preserved.

6.1. **Proposition.** *Let $x_0(M)$ be the embedding of a closed n-dimensional manifold M into $\mathbb{S}^{n+1}$, such that $x_0(M)$ is strictly convex. Let F be a positive, 1-homogeneous, strictly monotone and inverse concave[2] curvature function. Then along any solution x of (3.1) with initial embedding x_0 all flow hypersurfaces are strictly convex.*

Proof. Let b be the inverse of the Weingarten map, which exists at least for a short time. We show that for a smooth solution

$$x\colon [0,T^*) \times M \to \mathbb{S}^{n+1}$$

all M_t, $t < T^*$, are strictly convex. From

$$\dot{b}^k_m = -b^k_j \dot{h}^j_i b^i_m, \quad F^{qs} b^k_{m;qs} = 2F^{qs} b^k_j h^j_{p;q} b^p_r h^r_{i;s} b^i_m - F^{qs} b^k_p h^p_{l;qs} b^l_m,$$

$$u_{;i} = \lambda h^k_i r_{;k}$$

and (3.11) we deduce

$$\begin{aligned} \mathcal{L} b^k_m = {} & \frac{n}{F^2}\left(\frac{2}{F}F^{rs}F^{pq} - 2F^{qs}b^{pr} - F^{pq,rs}\right) b^k_j b^i_m h_{rs;i} h_{pq;}{}^j \\ & - \frac{\lambda^2}{\lambda'^2}\left(b^k_l r_;{}^l r_{;m} + r_;{}^k b^l_m r_{;l}\right) - \frac{2u\lambda^2}{\lambda'^3} b^k_j r_;{}^j b^i_m r_{;i} \\ & + \psi_1 b^k_m - \frac{2u}{\lambda'} b^k_l b^l_m + \psi_2 \delta^k_m, \end{aligned} \tag{6.1}$$

where ψ_i, $i = 1,2$ are some functions, which are bounded on every compact interval $[0, T_0] \subset [0, T^*)$. If the convexity is lost at some time $T_0 < T^*$, then the largest eigenvalue of b blows up at T_0. Although the largest eigenvalue is not a smooth function, we can still apply (6.1) to estimate it by using the following well known trick, compare e.g. the proof of [18, Lemma 6.1]:

Define

$$\phi = \sup\{b_{ij}\eta^i\eta^j \colon g_{ij}\eta^i\eta^j = 1\}$$

and suppose this function attains a maximum at (t_0, ξ_0), $t_0 < T_0$. Use normal coordinates around (t_0, ξ_0) with

$$g_{ij} = \delta_{ij}, \quad b_{ij} = \kappa_i^{-1}\delta_{ij}, \quad \kappa_1^{-1} \leq \cdots \leq \kappa_n^{-1}$$

at (t_0, ξ_0). Around (t_0, ξ_0) let η be the vector field

$$\eta = (0, \ldots, 0, 1)$$

and define

$$\tilde{\phi} = \frac{b_{ij}\eta^i\eta^j}{g_{ij}\eta^i\eta^j},$$

[2] $F(\kappa_1, \cdots, \kappa_n)$ is called inverse concave if $\tilde{F}(\kappa_1, \cdots, \kappa_n) = F^{-1}(\kappa_1^{-1}, \cdots, \kappa_n^{-1})$ is concave.

then locally around (t_0, ξ_0) we have $\tilde{\phi} \leq \phi$ and at this point there holds

$$\dot{\tilde{\phi}} = \dot{b}_{nn} - 2\mathcal{F} = \dot{b}^n_n$$

and the spatial derivatives also coincide. Thus at (t_0, ξ_0) the function $\tilde{\phi}$ and b^n_n satisfy the same evolution equation, hence it suffices to show that the right hand side of (6.1) is negative at the point (t_0, ξ_0).

The first line is negative due to the inverse concavity of F, compare the proof in [43, p. 112], while for the rest the good terms involving $b^k_l b^l_m$ are surely dominating. This completes the proof. □

7. Bounds on the speed and the curvature

In this section we deduce the remaining ingredients which are necessary to obtain longtime existence, namely we need a full bound on the second fundamental form and in turn, to apply the Krylov-Safonov theory, we need a lower bound on the curvature function to show that the operator $\mathcal{L}$ is uniformly parabolic along the flow. We start with the spherical case.

7.1. **The spherical case.**

7.1. **Lemma.** *Let $x_0(M)$ be the embedding of a closed n-dimensional manifold M into $\mathbb{S}^{n+1}$, such that $x_0(M)$ is strictly convex. Let*

$$F = n\frac{H_k}{H_{k-1}}.$$

Then along any solution x of (3.1) *with initial embedding x_0 there exists a constant c, such that*

$$\|A\|^2 \leq c.$$

Proof. Due to the convexity preservation, Proposition 6.1, it suffices to bound the mean curvature H from above. Note $u \geq c_0 > 0$ by Proposition 5.1 (we may also use the convexity to get this, cf. [20, Lemma 2.7.10]). We use the auxiliary function

$$w = \log H - \log u$$

and deduce from (3.5), (3.11), the concavity of F and

$$u_{;i} = \lambda h^k_i r_{;k},$$

that at a maximal point of w:

$$\begin{aligned} 0 \leq \mathcal{L}w &= \frac{1}{H}\mathcal{L}H - \frac{1}{u}\mathcal{L}u \\ &\leq c + \frac{c}{H} - \frac{2n}{FH}\|A\|^2 \\ &\leq c + \frac{c}{H} - \frac{2}{F}H. \end{aligned}$$

Since F is bounded from above by Proposition 4.1, we get a upper bound of H from above. □

We use the previous result to get bounds from below on F.

7.2. **Lemma.** *Under the assumptions of Lemma 7.1 there exists a positive constant c such that*

$$F \geq c.$$

Proof. We use the same method as in [36, Prop. 5.3] and bound the auxiliary function

$$z = -\log F + f(r),$$

where

$$f(r) = -\log\left(\lambda' - \alpha\right), \quad 0 < \alpha < \frac{1}{2}\lambda'(\sup r_0).$$

Since $\lambda'' = -\lambda$, it is direct to check that

$$1 - f'\frac{\lambda'}{\lambda} = -\frac{\alpha}{\lambda' - \alpha}, \quad f'^2 + f'\frac{\lambda'}{\lambda} - f'' = 0. \tag{7.1}$$

From the convexity, the 1-homogeneity of F and Lemma 7.1, we see

$$\frac{n}{F^2}F^{ij}h_{ik}h^k_j \le \frac{nH}{F} \le \frac{c}{F}. \tag{7.2}$$

Using (3.4), (3.6) and (7.2),

$$\begin{aligned}
\mathcal{L}z &= -\frac{1}{F}\mathcal{L}F - \frac{n}{F^4}F^{ij}F_{;i}F_{;j} + f'\mathcal{L}r - f''\frac{n}{F^2}F^{ij}r_{;i}r_{;j} \\
&\le \frac{n}{F^2}F^{ij}(\log F)_{;i}(\log F)_{;j} + \frac{c}{F} + c + \frac{n}{F^2}F^{ij}g_{ij} \\
&\quad + f'\frac{c}{F} - f'\frac{n\lambda'}{\lambda F^2}F^{ij}g_{ij} + f'\frac{n\lambda'}{\lambda F^2}F^{ij}r_{;i}r_{;j} - f''\frac{n}{F^2}F^{ij}r_{;i}r_{;j}.
\end{aligned}$$

At a maximal point of z, we use $(\log F)_{;i} = f'r_{;i}$ and (7.1) to obtain

$$\begin{aligned}
0 \le \mathcal{L}z &\le \frac{n}{F^2}F^{ij}r_{;i}r_{;j}\left(f'^2 + f'\frac{\lambda'}{\lambda} - f''\right) + \frac{n}{F^2}F^{ij}g_{ij}\left(1 - f'\frac{\lambda'}{\lambda}\right) + \frac{c}{F} + c + f'\frac{c}{F} \\
&= -\frac{\alpha}{\lambda' - \alpha}\frac{n}{F^2}F^{ij}g_{ij} + \frac{c}{F} + c \\
&< 0,
\end{aligned}$$

if F is small enough, since $F^{ij}g_{ij} \ge n$. ☐

Now we finish the a priori estimates in the spherical case.

7.3. **Proposition.** *Let $x_0(M)$ be the embedding of a closed n-dimensional manifold M into $\mathbb{S}^{n+1}$, such that $x_0(M)$ is strictly convex. Let*

$$F = n\frac{H_k}{H_{k-1}}.$$

Then any solution x of (3.1) *with initial embedding x_0 exists for all positive times with uniform C^∞-estimates.*

Proof. We have uniform C^2-bounds from Proposition 6.1 and Lemma 7.1. Due to Lemma 7.2 we know that the principal curvatures range within a compact subset of the domain on definition of F. Hence we have the uniform parabolicity of the operator $\mathcal{L}$. Due to the concavity of the operator, we can apply the regularity theory of Krylov and Safonov, [35], to deduce $C^{2,\alpha}$ bounds and in turn C^∞ bounds using the Schauder theory. Thus we can extend the flow beyond any finite T. ☐

7.2. **The general case.** We provide the bounds on the principal curvatures and on the curvature function from below in case of mild assumptions on the warping factor.

7.4. **Proposition.** *Let $a, b \in \mathbb{R}$ and $(N, \bar{g})$ be the warped space $((a,b)\times\mathbb{S}^n, dr^2+\lambda^2(r)\sigma)$ with $\lambda > 0$, $\lambda' > 0$ and $\lambda'' \geq 0$. Let $F \in C^\infty(\Gamma)$ be a positive 1-homogeneous, strictly monotone and concave curvature function and let $x_0(M)$ be the embedding of a closed n-dimensional manifold M into N, such that $x_0(M)$ is a graph over the domain $\mathbb{S}^n$ and such that $\kappa \in \Gamma$ for all n-tuples of principal curvatures along $x_0(M)$. Then along any solution x of* (3.1) *with initial embedding x_0 there exists a positive constant c, such that*

$$F \geq c.$$

7.5. *Remark.* Proposition 7.4 is the only place where we use $\lambda'' \geq 0$ for proving Theorem 1.3.

Proof. We deduce the evolution of the function $\partial_t\varphi$, where φ is defined as in (5.1). Recall that there holds (5.2),

$$\begin{aligned}\partial_t\varphi &= \frac{1}{\lambda}\left(\frac{n}{F\left(\frac{\lambda'}{\lambda v}\delta^j_i - \frac{1}{\lambda v}\tilde{g}^{jk}\varphi_{ki}\right)} - \frac{u}{\lambda'}\right)v\\ &= \frac{nv^2}{F(\lambda'\delta^j_i - \tilde{g}^{jk}\varphi_{ki})} - \frac{1}{\lambda'} =: G(\varphi, \hat{\nabla}\varphi, \hat{\nabla}^2\varphi),\end{aligned}$$

where $\tilde{g}^{ij} = \sigma^{ij} - \frac{\varphi^i\varphi^j}{v^2}$. Differentiation gives

$$\partial_t(\partial_t\varphi) = G^{ij}(\partial_t\varphi)_{ij} + G^{\varphi_p}(\partial_t\varphi)_p + G^\varphi\partial_t\varphi.$$

From (5.3) we obtain

$$G^\varphi \leq -\frac{n^2v^2\lambda''\lambda}{F^2} + \frac{\lambda''\lambda}{\lambda'^2} = -\frac{\lambda''\lambda}{v^2}\frac{n^2v^4}{F^2} + \frac{\lambda''\lambda}{\lambda'^2} = -\frac{\lambda''\lambda}{v^2}\left(\partial_t\varphi + \frac{1}{\lambda'}\right)^2 + \frac{\lambda''\lambda}{\lambda'^2}.$$

Since we already have $v \leq c$ due to Proposition 5.1, the third order leading term is dominating with a non-positive sign. The maximum principle gives an upper bound for $\partial_t\varphi$ and hence the result. ☐

7.6. **Proposition.** *Under the assumptions of Proposition 7.4 there exists a constant c, such that*

$$\|A\|^2 \leq c.$$

Proof. In applying the maximum principle to the evolution of (h^i_j) we proceed similarly to the proof of Proposition 6.1. Define

$$\phi = \sup\{h_{ij}\eta^i\eta^j : g_{ij}\eta^i\eta^j = 1\}$$

and suppose the function

$$w = \log\phi + f(u) + \alpha r$$

attains a maximum at (t_0, ξ_0), $t_0 < T_0$, where f is defined by

$$f(u) = -\log(u-\beta),$$

where $\beta = \frac{1}{2}\min u$. Note that

$$1 + f'u = \frac{-\beta}{u-\beta} < 0.$$

Using normal coordinates around (t_0, ξ_0) with

$$g_{ij} = \delta_{ij}, \quad h_{ij} = \kappa_i \delta_{ij}, \quad \kappa_1 \le \cdots \le \kappa_n,$$

and using (3.4), (3.5) and (3.8), we may pretend that the evolution equation of w at the point (t_0, ξ_0) is given by

$$\begin{aligned} \mathcal{L}w \le{}& \frac{n}{F^2}\frac{2}{\kappa_n - \kappa_1}\sum_{k=1}^{n}(F^{nn} - F^{kk})(h_{nk;n})^2 (h^n_n)^{-1} + c + \frac{c}{\kappa_n} + \frac{n}{F^2}F^{kl}h_{rk}h^r_l \\ &- \frac{2n}{F}\kappa_n + \frac{c(1+\kappa_n^{-1})}{F^2}F^{ij}g_{ij} + \frac{n}{F^2}F^{ij}(\log h^n_n)_{;i}(\log h^n_n)_{;j} \\ &+ \frac{n}{F^2}\left(F^{kl}h_{rk}h^r_l - \frac{1}{n}F^2\right) f' u - f' \frac{\lambda''\lambda}{\lambda'^2}\|\nabla r\|^2 u + c|f'| \\ &- f'' \frac{n}{F^2}F^{ij}u_{;i}u_{;j} + \frac{\alpha c}{F} - \alpha\frac{\lambda}{\lambda'} - \frac{n\alpha\lambda'}{\lambda F^2}F^{ij}(g_{ij} - r_{;i}r_{;j}), \end{aligned} \tag{7.3}$$

where we used a trick that already appeared in the proof of [14, Prop. 6.3] and in a similar fashion in [19, Thm. 9.7], namely that due to the concavity of F there holds

$$F^{kl,rs}\eta_{kl}\eta_{rs} \le \sum_{k\neq l}\frac{F^{kk} - F^{ll}}{\kappa_k - \kappa_l}\eta^2_{kl} \le \frac{2}{\kappa_n - \kappa_1}\sum_{k=1}^{n}(F^{nn} - F^{kk})\eta^2_{nk}$$

for all symmetric matrices (η_{kl}), cf. [20, Lemma 2.1.14]. Furthermore we have

$$F^{nn} \le \cdots \le F^{11},$$

cf. [13, Lemma 2]. In order to estimate (7.3), we distinguish two cases.

Case 1: $\kappa_1 < -\epsilon_1\kappa_n$, $0 < \epsilon_1 < \frac{1}{2}$. Then

$$F^{ij}h_{ik}h^k_j \ge F^{11}\kappa_1^2 \ge \frac{1}{n}F^{ij}g_{ij}\kappa_1^2 \ge \frac{1}{n}F^{ij}g_{ij}\epsilon_1^2\kappa_n^2.$$

We use $\nabla w = 0$ to estimate

$$\frac{n}{F^2}F^{ij}(\log h^n_n)_{;i}(\log h^n_n)_{;j} = f'^2\frac{n}{F^2}F^{ij}u_{;i}u_{;j} + f'\frac{2n\alpha}{F^2}F^{ij}u_{;i}r_{;j} + \frac{n\alpha^2}{F^2}F^{ij}r_{;i}r_{;j}.$$

If κ_n is sufficiently large, in this case (7.3) becomes

$$\begin{aligned} \mathcal{L}w \le{}& \frac{1}{F^2}F^{ij}g_{ij}(\epsilon_1^2\kappa_n^2(1 + f'u) + (c + |f'|\alpha)\kappa_n + c\alpha^2 + c) + c(|f'| + 1) \\ &- \frac{2n}{F}(\kappa_n - \alpha c) - \alpha\frac{\lambda}{\lambda'} - \frac{n}{F^2}F^{ij}u_{;i}u_{;j}(f'' - f'^2), \end{aligned}$$

which is negative for large κ_n, after fixing $\alpha_0 = \alpha_0(M_0, \sup r_0, \inf r_0, \lambda)$ large enough to ensure

$$c(|f'| + 1) - \alpha_0\frac{\lambda}{\lambda'} < 0.$$

We also use $1 + f'u \le c < 0$ and $f'' - f'^2 = 0$. Hence in this case any $\alpha \ge \alpha_0$ yields an upper bound for κ_n.

Case 2: $\kappa_1 \geq -\epsilon_1 \kappa_n$. Then

$$\begin{aligned}
&\frac{2}{\kappa_n - \kappa_1}\sum_{k=1}^{n}(F^{nn} - F^{kk})(h_{nk;n})^2 (h_n^n)^{-1} \\
&\leq \frac{2}{1+\epsilon_1}\sum_{k=1}^{n}(F^{nn} - F^{kk})(h_{nk;n})^2 (h_n^n)^{-2} \\
&\leq \frac{2}{1+\epsilon_1}\sum_{k=1}^{n}(F^{nn} - F^{kk})(h_{nn;k})^2 (h_n^n)^{-2} + c(\epsilon_1)\sum_{k=1}^{n}(F^{kk} - F^{nn})\kappa_n^{-2} \\
&\quad + \frac{4}{1+\epsilon_1}\sum_{k=1}^{n}(F^{nn} - F^{kk})h_{nn;k}\bar{R}_{\alpha\beta\gamma\delta}\nu^a x^{\beta}_{\ ;n} x^{\gamma}_{\ ;n} x^{\delta}_{\ ;k}(h_n^n)^{-2} \\
&\leq \frac{2}{1+2\epsilon_1}\sum_{k=1}^{n}(F^{nn} - F^{kk})(h_{nn;k})^2 (h_n^n)^{-2} + c(\epsilon_1)\sum_{k=1}^{n}(F^{kk} - F^{nn})\kappa_n^{-2},
\end{aligned}$$

where we used the Codazzi equation (2.5) and the Cauchy-Schwarz inequality. We deduce further:

$$\begin{aligned}
&F^{ij}(\log h_n^n)_{;i}(\log h_n^n)_{;j} + \frac{2}{\kappa_n - \kappa_1}\sum_{k=1}^{n}(F^{nn} - F^{kk})(h_{nk;n})^2 (h_n^n)^{-1} \\
&\leq \frac{2}{1+2\epsilon_1}\sum_{k=1}^{n}F^{nn}(\log h_n^n)^2_{;k} - \frac{1-2\epsilon_1}{1+2\epsilon_1}\sum_{k=1}^{n}F^{kk}(\log h_n^n)^2_{;k} + c(\epsilon_1)F^{ij}g_{ij}\kappa_n^{-2} \\
&\leq \sum_{k=1}^{n}F^{nn}(\log h_n^n)^2_{;k} + c(\epsilon_1)F^{ij}g_{ij}\kappa_n^{-2} \\
&= c(\epsilon_1)F^{ij}g_{ij}\kappa_n^{-2} + f'^2 F^{nn}\|\nabla u\|^2 + 2\alpha f' F^{nn}\langle\nabla u, \nabla r\rangle + \alpha^2 F^{nn}\|\nabla r\|^2.
\end{aligned}$$

We plug this into (7.3) and obtain for large κ_n:

$$\begin{aligned}
\mathcal{L}w &\leq c + \frac{n}{F^2}F^{nn}\kappa_n^2(1+f'u) - \frac{2n}{F}(\kappa_n - \alpha c) + \frac{1}{F^2}F^{ij}g_{ij}\left(c + c(\epsilon_1) - \frac{n\alpha\lambda'}{v^2\lambda}\right) \\
&\quad - f''\frac{n}{F^2}F^{ij}u_{;i}u_{;j} - \alpha\frac{\lambda}{\lambda'} + f'^2\frac{n}{F^2}F^{nn}\|\nabla u\|^2 + \frac{2n\alpha f'}{F^2}F^{nn}\langle\nabla u, \nabla r\rangle \\
&\quad + \frac{n\alpha^2}{F^2}F^{nn}\|\nabla r\|^2 \\
&\leq \frac{n}{F^2}F^{nn}\left(\kappa_n^2(1+f'u) + 2\alpha|f'|c\kappa_n + \alpha^2\|\nabla r\|^2\right) - \frac{2n}{F}(\kappa_n - \alpha c) \\
&\quad + c - \alpha\frac{\lambda}{\lambda'} + \frac{1}{F^2}F^{ij}g_{ij}\left(c + c(\epsilon_1) - \frac{n\alpha\lambda'}{v^2\lambda}\right) \\
&< 0
\end{aligned}$$

after possibly enlarging α even further (compared to case 1) and for large κ_n. This completes the proof. $\square$

As in Proposition 7.3 we conclude:

7.7. **Proposition.** *Under the assumptions of Theorem 1.3 the flow* (3.1) *exists for all times with uniform C^∞-estimates.*

8. Proofs of the main theorems

We give the final arguments to complete the proofs concerning the flow results and start with the spherical case.

Proof of Theorem 1.1. In order to complete the proof of Theorem 1.1 with the help of Proposition 7.3, all we have to show is that each subsequential limit is a sphere independent of the subsequence as $t \to \infty$.

The evolution of the weighted enclosed volume

$$V(t) = \int_{\Omega_t} \lambda' dN$$

is

$$\dot{V}(t) = \int_{M_t} \left(\frac{n\lambda'}{F} - u\right) d\mu_t \geq \int_{M_t} \left(\frac{n\lambda'}{H} - u\right) d\mu_t \geq 0.$$

The first inequality is due to the concavity of F which implies $F \leq H$, [20, Lemma 2.2.20] and the second one is due to Brendle's Heintze-Karcher type inequality, [6, equ. (4)]. That is, V is increasing. Since V is obviously bounded we have

$$\int_0^\infty \int_{M_t} \left(\frac{n\lambda'}{H} - u\right) \, d\mu_t dt < \infty$$

and hence

$$\int_{M_t} \left(\frac{n\lambda'}{H} - u\right) \, d\mu_t \to 0.$$

So any convergent subsequence of M_t must converge to a sphere, due to the characterization of the limiting case in the Heintze-Karcher inequality. Due to the spherical barriers this sphere is unique and we conclude the proof of the theorem. $\square$

Now we turn to the other case and prove Theorem 1.3.

Proof of Theorem 1.3. Again it suffices to prove that there exists a subsequence that converges to a sphere. If no subsequence converges to a geodesic sphere then there can not be any subsequence for which $\|\nabla r\| \to 0$. Hence there exists a positive constant c such that for all times $t > 0$ we have

$$\max_{M_t} \|\nabla r\|^2 \geq c. \tag{8.1}$$

The area evolves according to

$$\frac{d}{dt}|M_t| = \int_{M_t} \mathcal{F}H \geq \int_{M_t} \left(n - \frac{Hu}{\lambda'}\right) = \int_{M_t} \frac{\operatorname{div}(\lambda\nabla r)}{\lambda'} = \int_{M_t} \frac{\lambda''\lambda}{\lambda'^2}\|\nabla r\|^2 \geq 0. \tag{8.2}$$

The inequality in (8.2) is again due to $F \leq H$. The last two equalities in (8.2) follow from the fact $\operatorname{div}(\lambda\nabla r) = n\lambda' - Hu$ and integration by parts respectively.

Due to the C^1-estimates the area is bounded and hence, because of $\lambda'' \geq 0$, every subsequential limit $M_t \to \tilde{M}$ must satisfy

$$\int_{\tilde{M}} \frac{\lambda''\lambda}{\lambda'^2}\|\nabla r\|^2 = 0,$$

whence

$$\lambda''\|\nabla r\|^2 = 0 \tag{8.3}$$

throughout any subsequential limit. For all $t > 0$, let

$$\xi_t := \operatorname{argmax}_{M_t} \|\nabla r\|^2.$$

We obtain that

$$\lambda''(\xi_t) \to 0, \quad t \to \infty,$$

for otherwise we reach a contradiction to (8.1) and (8.3). From (5.6) we obtain at the points (t, ξ_t),

$$\begin{aligned}
\mathcal{L}|\hat{\nabla}\varphi|^2 &\leq -2G^{ij}\sigma_{ij}|\hat{\nabla}\varphi|^2 + 2G^{ij}\varphi_i\varphi_j + 2G^{\varphi}|\hat{\nabla}\varphi|^2 \\
&\leq -\frac{2nv^2}{F^2}F^i_k\tilde{g}^{kj}\sigma_{ij}|\hat{\nabla}\varphi|^2 + \frac{2nv^2}{F^2}F^i_k\tilde{g}^{kj}\varphi_i\varphi_j + c\lambda''|\hat{\nabla}\varphi|^2 \\
&\leq -\epsilon|\hat{\nabla}\varphi|^2,
\end{aligned}$$

for some suitable $\epsilon > 0$. Thus $|\hat{\nabla}\varphi|^2$ actually has to decay exponentially and we obtain a contradiction to (8.1). □

9. Geometric inequalities

In this section we complete the proof of the geometric inequalities. First of all, along the flow $\frac{d}{dt}x = \mathcal{F}\nu$, we have the following variational formulas.

9.1. **Proposition.** *Let $M_t \subset N$ be a family of closed hypersurfaces evolving by $\frac{d}{dt}x = \mathcal{F}\nu$. Denote by Ω_t the enclosed domain by M_t and $\{a\} \times \mathbb{S}^n$. Then*

$$\frac{d}{dt}\int_{\Omega_t} f = \int_{M_t} f\mathcal{F} \quad \forall f \in C^\infty(N), \tag{9.1}$$

and

$$\frac{d}{dt}|M_t| = \int_{M_t} H\mathcal{F}. \tag{9.2}$$

If $\bar{\Delta}\lambda'\bar{g} - \bar{\nabla}^2\lambda' + \lambda'\overline{\mathrm{Rc}} = 0$, then

$$\frac{d}{dt}\int_{M_t} H\lambda' = \int_{M_t} (2\sigma_2\lambda' + 2H\left\langle\bar{\nabla}\lambda', \nu\right\rangle)\mathcal{F}. \tag{9.3}$$

Proof. The first and second ones are well known and have already been used in section 8. We compute the third one.

$$\begin{aligned}
\frac{d}{dt}\int_{M_t} H\lambda' &= \int_{M_t} \lambda'(-\Delta\mathcal{F} - \mathcal{F}|A|^2 - \mathcal{F}\overline{\mathrm{Rc}}(\nu,\nu)) \\
&\quad + \int_{M_t} \left(H\left\langle\bar{\nabla}\lambda', \nu\right\rangle + H^2\lambda'\right)\mathcal{F} \\
&= \int_{M_t} -(\bar{\Delta}\lambda' - \bar{\nabla}^2\lambda'(\nu,\nu) - H\left\langle\bar{\nabla}\lambda', \nu\right\rangle + \lambda'\overline{\mathrm{Rc}}(\nu,\nu))\mathcal{F} \\
&\quad + \int_{M_t} \left(H\left\langle\bar{\nabla}\lambda', \nu\right\rangle + (H^2 - |A|^2)\lambda'\right)\mathcal{F} \\
&= \int_{M_t} 2\sigma_2\lambda'\mathcal{F} + 2H\left\langle\bar{\nabla}\lambda', \nu\right\rangle\mathcal{F}.
\end{aligned}$$

□

9.2. **Proposition.** *Let $\Sigma \subset N$ be a closed hypersurface. If Σ is star-shaped and $\frac{\lambda''}{\lambda} + \frac{1-\lambda'^2}{\lambda^2} \geq 0$, then*

$$\int_\Sigma (n-1)H\lambda' \leq \int_\Sigma 2\sigma_2 u. \tag{9.4}$$

Proof. Multiplying σ_2^{ij} to (2.8), summing over i, j, integrating over Σ and using

$$\nabla_i \sigma_2^{ij} = h^i_{i;m} g^{mj} - h^{ij}{}_{;i} = -\overline{\mathrm{Rc}}(\nu, x_{;m}) g^{mj},$$

we have

$$\begin{aligned}
\int_\Sigma (n-1)H\lambda' - 2\sigma_2 u &= \int_\Sigma \sigma_2^{ij}(\lambda r_{;j})_{;i} \\
&= \int_\Sigma \lambda \overline{\mathrm{Rc}}(\nu, x_{;m}) r_;{}^m \\
&= \int_\Sigma -(n-1)\left[\frac{\lambda''}{\lambda} + \frac{1-\lambda'^2}{\lambda^2}\right] \lambda \|\nabla r\|^2 \langle \partial_r, \nu \rangle \\
&\le 0,
\end{aligned}$$

where we used the starshapedness and (2.7). ☐

Now we choose the flow as

$$\frac{d}{dt}x = \left(\frac{n}{H} - \frac{u}{\lambda'}\right)\nu. \tag{9.5}$$

Along this flow the area $|M_t|$ is non-decreasing and the quantity

$$\int_{M_t} H\lambda' d\mu_t - 2n \int_{\Omega_t} \frac{\lambda'\lambda''}{\lambda} dN$$

is non-increasing.

9.3. **Proposition.** *Under the assumptions of Theorem 1.5, let $M_t \subset N$ be a family of closed star-shaped hypersurfaces evolving by* (9.5)*. Then*

$$\frac{d}{dt}\int_{\Omega_t} \lambda' dN \ge 0, \tag{9.6}$$

$$\frac{d}{dt}|M_t| \ge 0 \tag{9.7}$$

and

$$\frac{d}{dt}\left(\int_{M_t} H\lambda' d\mu_t - 2n\int_{\Omega_t} \frac{\lambda'\lambda''}{\lambda} dN\right) \le 0. \tag{9.8}$$

Proof. We first note that all the assumptions in Proposition 9.1 and Proposition 9.2 are satisfied by the anti-de-Sitter Schwarzschild space and the hyperbolic space. Also the Heintze-Karcher type inequality holds for the anti-de-Sitter Schwarzschild space and the hyperbolic space. Thus inequality (9.6) is proved in the same way as the proof of Theorem 1.1 in section 8. Inequality (9.7) was proved in the proof of Theorem 1.3 in section 8. Next we show

(9.8). From (9.3) and (9.1), we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_{M_t} H\lambda' - 2n\int_{\Omega_t}\frac{\lambda'\lambda''}{\lambda}\right)\\
&= \int_{M_t}\left(2\sigma_2\lambda' + 2H\left\langle\bar\nabla\lambda',\nu\right\rangle - 2n\frac{\lambda'\lambda''}{\lambda}\right)\left(\frac{n}{H}-\frac{u}{\lambda'}\right)\\
&= \int_{M_t}2\sigma_2\lambda'\left(\frac{n}{H}-\frac{u}{\lambda'}\right) + \int_{M_t}\frac{\lambda''}{\lambda}\left(2Hu-2n\lambda'\right)\left(\frac{n}{H}-\frac{u}{\lambda'}\right) \qquad (9.9)\\
&\le \int_{M_t}\left((n-1)H\lambda' - 2\sigma_2 u\right) - \int_{M_t}2H\frac{\lambda'\lambda''}{\lambda}\left(\frac{n}{H}-\frac{u}{\lambda'}\right)^2\\
&\le 0.
\end{aligned}
$$

In the second equality, we used $\left\langle\bar\nabla\lambda',\nu\right\rangle = \frac{\lambda''}{\lambda}u$ and in the last two inequalities we used Newton-Maclaurin inequality, (9.4) and $\lambda'' \ge 0$. □

The inequalities in Theorem 1.5 follow immediately from the monotonicity in Proposition 9.3 and the convergence result of the flow. The classification of the equality case follows easily by checking the equality in (9.9). □

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Appendix A6

THE INVERSE MEAN CURVATURE FLOW PERPENDICULAR TO THE SPHERE

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THE INVERSE MEAN CURVATURE FLOW PERPENDICULAR TO THE SPHERE

BEN LAMBERT AND JULIAN SCHEUER

ABSTRACT. We consider the smooth inverse mean curvature flow of strictly convex hypersurfaces with boundary embedded in $\mathbb{R}^{n+1}$, which are perpendicular to the unit sphere from the inside. We prove that the flow hypersurfaces converge to the embedding of a flat disk in the norm of $C^{1,\beta}$, $\beta < 1$.

1. INTRODUCTION

We consider the inverse mean curvature flow in $\mathbb{R}^{n+1}$ with a Neumann boundary condition in a sphere. Let $\mathbb{D} = \mathbb{D}^n$ be the closed n-dimensional unit disk and $\tilde{N}$ be the outward unit normal of the inclusion $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$. Then we consider a family of embeddings

$$X\colon [0, T^*) \times \mathbb{D} \hookrightarrow \mathbb{R}^{n+1} \tag{1.1}$$

with a normal vector field N, the choice of which will be specified in a natural manner later, such that

$$\dot{X} = \frac{1}{H}N, \tag{1.2a}$$

$$X(\partial\mathbb{D}) = \partial X(\mathbb{D}) \subset \mathbb{S}^n, \tag{1.2b}$$

$$0 = \langle N_{|\partial\mathbb{D}}, \tilde{N}(X_{|\partial\mathbb{D}})\rangle, \tag{1.2c}$$

$$\langle\dot{\gamma}(0), \tilde{N}\rangle \geq 0 \quad \forall\gamma \in C^1((-\epsilon, 0], X(t, \mathbb{D}))\colon \gamma(0) \in \partial X(t, \mathbb{D}). \tag{1.2d}$$

We prove the following result.

1.1. **Theorem.** *Let*

$$X_0\colon \mathbb{D} \hookrightarrow M_0 \subset \mathbb{R}^{n+1} \tag{1.3}$$

be the embedding of a smooth and strictly convex hypersurface with normal vector field N_0, such that

$$X_0(\partial\mathbb{D}) \subset \mathbb{S}^n, \tag{1.4a}$$

$$\langle\dot{\gamma}(0), \tilde{N}\rangle \geq 0 \quad \forall\gamma \in C^1((-\epsilon, 0], M_0)\colon \gamma(0) \in \partial X_0(\mathbb{D}), \tag{1.4b}$$

$$\langle N_{0|\partial\mathbb{D}}, \tilde{N}_{|\partial\mathbb{D}}\rangle = 0. \tag{1.4c}$$

Then there exists a finite time $T^* < \infty$, $\alpha > 0$ and a unique solution

$$X \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T^*) \times \mathbb{D}) \cap C^\infty((0, T^*) \times \mathbb{D}, \mathbb{R}^{n+1}) \tag{1.5}$$

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of (1.2) *with initial hypersurface* M_0*, such that the embeddings* X_t *converge to the embedding of a flat unit disk as* $t \to T^*$*, in the sense that the height of the* $M_t = X(t, \mathbb{D})$ *over this disk converges to* 0.

1.2. *Remark.* The norm of convergence of the M_t to the disk will be specified in Remark 7.4, when we will have developed a suitable coordinate system to describe the M_t.

Our motivation for treating this problem arises from several directions. First of all, the inverse mean curvature flow (IMCF) has proven to be a useful tool in the theory of geometric inequalities, cf. [11] for the probably most famous result in this direction. The works which describe the asymptotic behaviour of the IMCF in Euclidean space include [4] and [21], whereas in the hyperbolic space we refer to [7] and [17]. For the IMCF of hypersurfaces of the sphere compare [8] and [14]. Those works deal with closed hypersurfaces.

Few years ago, the Ph.D. thesis [15] written by Thomas Marquardt appeared, also cf. [16]. Here the IMCF of hypersurfaces with boundary was considered and the embedded flowing hypersurfaces were supposed to be perpendicular to a convex cone in $\mathbb{R}^{n+1}$. However, short-time existence was derived in a much more general situation, in other ambient spaces and with other supporting hypersurfaces besides the cone. It appears to be a natural question, whether one can also obtain nice convergence results if one imposes perpendicularity to other hypersurfaces. Inspired by a recent result about rigidity of hypersurfaces in the sphere by Matthias Makowski and the second author, cf. [14], Oliver Schnürer suggested to the authors that this rigidity result might be helpful to consider the IMCF for hypersurfaces which are perpendicular to the sphere. Indeed, we were able to prove his conjecture that this flow must drive strictly convex hypersurfaces into the embedding of a disk.

The equivalent problem for the mean curvature flow was treated by Axel Stahl in [19] and [18], in which the flow was shown to contract to a point. Other choices of boundary manifolds for a graphical mean curvature flow have shown convergence of the flow to flat disks, see for example [12] and [10], as well as [9] for a levelset approach.

The proof of Theorem 1.1 is ordered as follows: In section 2 we agree on notation and in section 3 we collect the relevant evolution equations and boundary derivatives. In section 4 we make height and gradient estimates for convex hypersurfaces perpendicular to the sphere, which is of interest independently. In particular there follows that if the boundary of a convex manifold is contained in a hemisphere, then we have a lower height bound on the manifold. In section 5 we show that the flow may be written graphically. In section 6 we use the results of section 4 to demonstrate the two key estimates which in conjunction with rigidity results of [14] give the theorem. The first of these is that while the boundary stays away from an equator, a convex flow has a lower bound on H. The second shows that the flow remains convex up until the singular time. Therefore, due to rigidity at the boundary, ∂M_t must flow to an equator and so M_t must flow to a flat disk assuming that the flow may be suitably extended. In section 7 we clarify the necessary PDE existence results and show $C^{1,\beta}$ convergence. In the appendix we indicate how counterexamples to our general convergence result can be constructed, when the supporting hypersurface is not a sphere.

2. Setting and notation

There are various embeddings involved in (1.2), where in this section we suppress the time parameter for better readability, namely the inclusion

$$x\colon \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}, \tag{2.1}$$

the flow embeddings of the form

$$X\colon \mathbb{D} \hookrightarrow \mathbb{R}^{n+1}, \tag{2.2}$$

the inclusion

$$z\colon \partial\mathbb{D} \hookrightarrow \mathbb{D}, \tag{2.3}$$

as well as the derived embedding

$$y\colon \partial\mathbb{D} \hookrightarrow \mathbb{S}^n \tag{2.4}$$

satisfying

$$X \circ z = x \circ y. \tag{2.5}$$

Throughout this paper, we stick to the coordinate based notation for tensors.

Geometric quantities in $\mathbb{R}^{n+1}$ are denoted by a bar, e.g. $(\bar{g}_{\alpha\beta})$ for the Euclidean metric, where greek indices range from 0 to n. We will also write $\langle\cdot,\cdot\rangle$ for the Euclidean scalar product.

Geometric quantities in $\mathbb{S}^n$ are denoted by a check, e.g. $(\check{g}_{ij})$ for the induced metric of the embedding x, where latin indices range from 1 to n.

Induced quantities of embeddings $\mathbb{D} \hookrightarrow \mathbb{R}^{n+1}$ are denoted by latin letters, e.g. the embeddings X induce metrics (g_{ij}), normal vector fields N and second fundamental forms (h_{ij}), such that we have the Gaussian formula

$$X^\alpha_{ij} = -h_{ij}N^\alpha. \tag{2.6}$$

A hypersurface $M \hookrightarrow \mathbb{R}^{n+1}$ is called strictly convex, if N can smoothly be chosen, such that (h_{ij}) is positive definite. For a strictly convex hypersurface we will choose N like this.

Induced quantities of embeddings to $\partial\mathbb{D} \hookrightarrow \mathbb{S}^n$ are denoted by greek letters, e.g. the embeddings y induce metrics (γ_{IJ}), normal vector fields ν and second fundamental forms (η_{IJ}), where capital latin indices range from 2 to n.

Coordinate systems in $\partial\mathbb{D}$ will be denoted by (ξ^I), $2 \leq I \leq n$.

Define H to be the mean curvature of the embeddings X,

$$H = g^{ij}h_{ij}, \tag{2.7}$$

where (g^{ij}) is the inverse of (g_{ij}).

For an embedded manifold $M^n \hookrightarrow N^{n+1}$ and a function $u\colon M \to \mathbb{R}$, covariant derivatives with respect to the induced metric are denoted by indices, e.g. u_{ij}. If ambiguities are possible, e.g. in the case of tensor derivation, covariant derivatives are denoted by a semicolon, e.g. $h_{ij;k}$. Standard partial derivatives are denoted by a comma, e.g. $u_{i,j}$.

3. Evolution equations and boundary derivatives

For the inverse mean curvature flow the interior evolution equations are well-known. We need the spatial boundary derivatives of various curvature quantities, when the supporting hypersurface is a sphere. The calculations are quite similar to those in [15] and [18]. For the sake of completeness and for a better comprehensibility of the different notation, let us derive them in detail.

3.1. *Remark.* Short-time existence for the flow (1.2) was derived in [15, Thm. 2.12]. Thus we are justified to use (1.2) to calculate the boundary derivatives.

3.2. *Remark.* Due to (1.2c) we obtain that

$$\tilde{N} \in X_*(T\mathbb{D}) \tag{3.1}$$

and thus at boundary points there holds

$$\tilde{n} \equiv (\langle X_k, \tilde{N}\rangle) \in T^{0,1}\mathbb{D}. \tag{3.2}$$

Thus, using (2.5), we see that

$$\mathcal{B} = (\tilde{n}, z_2, \ldots, z_n) \tag{3.3}$$

forms a basis of $T_y\mathbb{D}$ for all $y \in \partial\mathbb{D}$. Here we slightly abuse notation and let $\tilde{n}$ denote the contravariant version of $\tilde{n}$ as well. Furthermore we have

$$g_{ij}\tilde{n}^i z_I^j = 0, \quad 2 \le I \le n. \tag{3.4}$$

Boundary derivatives.

3.3. **Lemma.** *On $\partial\mathbb{D}$ there holds*

$$H_i\tilde{n}^i = -H. \tag{3.5}$$

Proof. Note that from

$$\dot{X} = \frac{1}{H}N, \tag{3.6}$$

which also holds on $\partial\mathbb{D}$, we obtain from (2.5) that

$$\frac{1}{H}x_i\nu^i = \frac{1}{H}N = \frac{d}{dt}(X \circ z) = x_i\dot{y}^i, \tag{3.7}$$

where ν denotes the pullback of N along x, which is well defined by (1.2c). We obtain that

$$\dot{y} = \frac{1}{H}\nu \tag{3.8}$$

holds in $T\mathbb{S}^n$. Differentiating (1.2c) with respect to time we obtain

$$\begin{aligned} 0 &= \langle \dot{N}, \tilde{N}\rangle + \langle N, \tilde{N}_i\dot{y}^i\rangle \\ &= \frac{1}{H^2}\langle X_i H^i, \tilde{N}\rangle + \frac{1}{H}\langle N, \check{h}_i^k x_k \nu^i\rangle, \end{aligned} \tag{3.9}$$

which implies the result in view of $\check{h}_i^k = \delta_i^k$. ☐ ☐

3.4. **Lemma.** *On $\partial\mathbb{D}$ there hold*

(i) $h_{ij}\tilde{n}^i z_I^j = 0, \quad 2 \le I \le n,$

(ii) $h_{ij;k} z_I^i z_J^j \tilde{n}^k = -h_{ij} z_I^i z_J^j + h_{ij}\tilde{n}^i\tilde{n}^j g_{kl} z_I^k z_J^l.$

Proof. Differentiating (1.2c) with respect to ξ^I yields, also using (2.5),

$$\begin{aligned} 0 &= \langle N_I, \tilde{N}\rangle + \langle N, \tilde{N}_I\rangle \\ &= h_l^k \tilde{n}_k z_I^l + \langle N, \check{h}_l^k x_k y_I^l\rangle \\ &= h_{ij}\tilde{n}^i z_I^j. \end{aligned} \tag{3.10}$$

Differentiate (2.5) twice to obtain

$$X_{kl} z_I^k z_J^l + X_k z_{IJ}^k = x_k y_{IJ}^k + x_{kl} y_I^k y_J^l. \tag{3.11}$$

Note that the first term on each of the two sides of (3.11) is a multiple of N due to the Gaussian formula in both cases. Thus by taking the scalar product with X_k we obtain

$$z_{IJ}^k = -\check{h}_{lm}\tilde{n}^k y_I^l y_J^m = -\gamma_{IJ}\tilde{n}^k = -g_{ij} z_I^i z_J^j \tilde{n}^k, \tag{3.12}$$

where we used that $\check{h}_{ij} = \check{g}_{ij} = \bar{g}_{\alpha\beta} x_i^\alpha x_j^\beta$ and (2.5).

Differentiating (3.10) with respect to ξ^J yields

$$\begin{aligned} h_{ij;k} z_J^k \tilde{n}^i z_I^j &= -h_{ij}\tilde{n}_J^i z_I^j - h_{ij}\tilde{n}^i z_{IJ}^j \\ &= -h_{ij} z_J^i z_I^j + h_{ij}\tilde{n}^i\tilde{n}^j\gamma_{IJ}, \end{aligned} \tag{3.13}$$

where we used

$$\begin{aligned} \tilde{n}_J^i &= (g^{ki}\bar{g}_{\alpha\beta} X_k^\alpha \tilde{N}^\beta)_{;J} \\ &= g^{ki}\bar{g}_{\alpha\beta} X_{kl}^\alpha \tilde{N}^\beta z_J^l + g^{ki}\bar{g}_{\alpha\beta} X_k^\alpha \tilde{N}_l^\beta y_J^l \\ &= g^{ki}\bar{g}_{\alpha\beta}\delta_l^m x_m^\beta y_J^l X_k^\alpha \\ &= z_J^i \end{aligned} \tag{3.14}$$

to transform the first term and (3.12) to transform the second term. ☐ ☐

3.5. **Lemma.** *On $\partial\mathbb{D}$ there holds*

$$h_{ij;k}\tilde{n}^i\tilde{n}^j\tilde{n}^k = -n h_{ij}\tilde{n}^i\tilde{n}^j. \tag{3.15}$$

Proof. With respect to the basis $\mathcal{B}$, g and A split, compare Remark 3.2 and Lemma 3.4. Therefore we have

$$\gamma^{IJ} z_I^i z_J^j = g^{ij} - \tilde{n}^i\tilde{n}^j \tag{3.16}$$

and thus

$$\begin{aligned} -H = H_k\tilde{n}^k &= g^{ij} h_{ij;k}\tilde{n}^k \\ &= h_{ij;k}\tilde{n}^i\tilde{n}^j\tilde{n}^k + h_{ij;k} z_I^i z_J^j \tilde{n}^k\gamma^{IJ} \\ &= h_{ij;k}\tilde{n}^i\tilde{n}^j\tilde{n}^k - h_{ij} z_I^i z_J^j\gamma^{IJ} + h_{ij}\tilde{n}^i\tilde{n}^j g_{kl} z_I^k z_J^l\gamma^{IJ} \\ &= h_{ij;k}\tilde{n}^i\tilde{n}^j\tilde{n}^k - H + n h_{ij}\tilde{n}^i\tilde{n}^j. \end{aligned} \tag{3.17}$$

☐ ☐

We need another lemma about the induced embedding.

3.6. **Lemma.** *The second fundamental form* (η_{IJ}) *with respect to the normal* $-\nu$ *as in* (3.8) *of the induced embedding*

$$y\colon \partial\mathbb{D} \hookrightarrow \mathbb{S}^n \tag{3.18}$$

satisfies

$$\eta_{IJ} = h_{kl} z^k_I z^l_J. \tag{3.19}$$

In particular, if X *is the embedding of a convex hypersurface into* $\mathbb{R}^{n+1}$, y *is the embedding of a convex hypersurface into the sphere* $\mathbb{S}^n$.

Proof. Differentiating (2.5) twice, we obtain from (3.12)

$$\begin{aligned} -x_k \eta_{IJ} \nu^k &= -h_{kl} z^k_I z^l_J N + X_k z^k_{IJ} + \gamma_{IJ} \tilde{N} \\ &= -h_{kl} z^k_I z^l_J N. \end{aligned} \tag{3.20}$$

□ □

To understand how the height of our hypersurfaces over a hyperplane behaves, we have the following lemma.

3.7. **Lemma.** *Let*

$$X_0\colon \mathbb{D} \to M_0 \hookrightarrow \mathbb{R}^{n+1} \tag{3.21}$$

be an embedding as in (1.4). *Let* $\omega \in \mathbb{R}^{n+1}$. *Then the height over the hyperplane* $\omega^\perp$,

$$w = \langle X, \omega \rangle, \tag{3.22}$$

satisfies

$$w_k \tilde{n}^k = w \tag{3.23}$$

on $\partial\mathbb{D}$. *In particular, if* ω *is chosen, such that* w *is positive on* $\partial\mathbb{D}$, w *attains its global minimum in the interior of* $\mathbb{D}$.

Proof. On $\partial\mathbb{D}$ we have

$$\begin{aligned} w_k \tilde{n}^k &= \bar{g}_{\alpha\beta} X^\alpha_k \omega^\beta g^{kl} \bar{g}_{\gamma\delta} X^\gamma_l \tilde{N}^\delta \\ &= \bar{g}_{\beta\delta} \omega^\beta \tilde{N}^\delta \\ &= \langle \tilde{N}, \omega \rangle \\ &= w, \end{aligned} \tag{3.24}$$

since on the boundary X maps into $\mathbb{S}^n$ and here the position vector X equals the outer normal $\tilde{N}$. □ □

Evolution equations. We need the following evolution equations.

3.8. **Lemma.** *The speed*

$$\Phi = -\frac{1}{H} \tag{3.25}$$

satisfies

$$\dot{\Phi} - \frac{1}{H^2} \Delta\Phi = \frac{\|A\|^2}{H^2} \Phi \tag{3.26}$$

in the interior and

$$\Phi_k \tilde{n}^k = \Phi \tag{3.27}$$

on the boundary.

Proof. The interior equation follows from [6, Lemma 2.3.4] and the boundary derivative from Lemma 3.3. □ □

3.9. **Lemma.** *Let* $\omega \in \mathbb{R}^{n+1}$. *Then the height*

$$w = \langle X, \omega \rangle \tag{3.28}$$

of M_t *over the plane* $\omega^\perp$ *satisfies*

$$\dot{w} - \frac{1}{H^2}\Delta w = \frac{2}{H}\langle N, \omega \rangle \tag{3.29}$$

in the interior and

$$w_k \tilde{n}^k = w \tag{3.30}$$

on the boundary.

Proof. The interior equation comes from (1.2a) and the boundary derivative is derived in Lemma 3.7. □ □

Applying a strictly convex function in $\mathbb{R}^{n+1}$ to X yields a very useful evolution equation, the derivation of which is a simple calculation.

3.10. **Lemma.** *Let* $\chi \in C^2(\mathbb{R}^{n+1})$. *Then* $\chi = \chi(X)$ *satisfies*

$$\dot{\chi} - \frac{1}{H^2}\Delta \chi = \frac{2}{H}\chi_\alpha N^\alpha - \frac{1}{H^2}\chi_{\alpha\beta} X_i^\alpha X_j^\beta g^{ij} \tag{3.31}$$

in the interior and

$$\chi_i \tilde{n}^i = \langle D\chi, \tilde{N} \rangle \tag{3.32}$$

on the boundary.

4. Height estimates

4.1. **Definition.** (i) For a convex hypersurface M_0 satisfying (1.4) let $\operatorname{conv}(\partial M_0)$ denote the closed convex body in the sphere enclosed by the convex hypersurface $\partial M_0 \hookrightarrow \mathbb{S}^n$, cf. Lemma 3.6.

(ii) For a point $x_0 \in \mathbb{S}^n$, $\mathcal{H}(x_0)$ denotes the closed hemisphere in $\mathbb{S}^n$ with center x_0. The corresponding equator is denoted by $\mathcal{S}(x_0)$.

4.2. **Lemma.** *Let* M_0 *be a convex hypersurface satisfying* (1.4) *and*

$$C_0 = \{x \in \mathbb{R}^{n+1} \colon x = sp, s \geq 0, p \in \operatorname{conv}(\partial M_0)\}. \tag{4.1}$$

Then there holds

$$M_0 \subset C_0. \tag{4.2}$$

Proof. C_0 is a convex cone in $\mathbb{R}^{n+1}$, cf. [3, Prop. 2], and is made of an intersection of half-spaces in $\mathbb{R}^{n+1}$ with normal N_0,

$$C_0 = \bigcap_{y\in\partial M_0} \{x \in \mathbb{R}^{n+1} \colon \langle x - y, N_0\rangle \le 0\}. \tag{4.3}$$

The tangent spaces of C_0 and M_0 coincide at all boundary points due to (1.4c) and hence for all boundary points y, M_0 lies on the same side of the tangent plane T_yM_0 as C_0. □ □

In the sequel we need the following simple geometric lemma.

4.3. **Lemma.** *Let $R > 0$, $e_0 \in \mathbb{R}^{n+1}$ be a unit vector and $C \subset \mathbb{R}^{n+1}$ be a convex closed cone. Then for all $\epsilon > 0$ there exists $\delta > 0$, such that*

$$\langle a, e_0\rangle \ge \cos\left(\frac{\pi}{2} - \epsilon\right)\|a\| \quad \forall a \in C \tag{4.4}$$

implies

$$\langle x, e_0\rangle \ge R + \delta \quad \forall B_R(x) \subset C. \tag{4.5}$$

Proof. Suppose the claim was false. Then there existed $\epsilon > 0$ and a sequence of Euclidean balls $B_R(x_k) \subset C$ with the property

$$R \le \langle x_k, e_0\rangle < R + \frac{1}{k} \tag{4.6}$$

and such that (4.4) holds. Without loss of generality assume that x_k converges to some $x \in C$. Then we also have

$$B_R(x) \subset C, \tag{4.7}$$

since C is closed. Then

$$a = x - Re_0 \in \bar{B}_R(x) \tag{4.8}$$

and due to (4.4) there holds $a = 0$. Thus we have

$$x = (R, 0, \dots, 0) \tag{4.9}$$

and hence a contradiction to (4.7), since C does not hit $\{x^0 = 0\}$ tangentially at 0. □ □

4.4. **Lemma.** *Let*

$$X_0 \colon M \hookrightarrow \mathbb{R}^{n+1} \tag{4.10}$$

be the embedding of a strictly convex hypersurface M_0, such that (1.4) holds. Let $e_0 \in$ $\operatorname{int}(\operatorname{conv}(\partial M_0))$ be a direction, such that $\operatorname{conv}(\partial M_0)$ is contained in the open hemisphere $\operatorname{int}(\mathcal{H}(e_0))$. Then we have

$$\varphi := \langle N_0, e_0\rangle \le c_0 \tag{4.11}$$

for some constant $c_0 < 0$, which only depends on the distance of e_0 to ∂M_0.

Proof. The Gauss map of the embedding X_0,

$$N_0 \colon \mathbb{D} \hookrightarrow \mathbb{S}^n, \tag{4.12}$$

is a diffeomorphism onto its image due to the strict convexity. By Lemma 3.6 and [6, Thm. 9.2.5] the restriction

$$N_{0|\partial\mathbb{D}} \colon \mathbb{S}^{n-1} \hookrightarrow \mathbb{S}^n \tag{4.13}$$

is a convex embedding and by [2, Thm. 1.1], there exist two disjoint open connected components A and B, such that

$$\mathbb{S}^n \backslash N_0(\partial\mathbb{D}) = A \cup B \tag{4.14}$$

and A is the interior of the strictly convex body in the sphere, which $N_0(\partial\mathbb{D})$ bounds. Since $\operatorname{conv}(\partial M_0)$ is chosen to be contained in $\mathcal{H}(e_0)$, we have

$$\partial A \subset \mathcal{H}(-e_0) \tag{4.15}$$

and from [6, Thm. 9.2.9, Thm. 9.2.10] we obtain

$$-e_0 \in A \subset \bar{A} \subset \mathcal{H}(-e_0). \tag{4.16}$$

We have either

$$N_0(\mathbb{D}\backslash\partial\mathbb{D}) \subset A \tag{4.17}$$

or

$$N_0(\mathbb{D}\backslash\partial\mathbb{D}) \subset B, \tag{4.18}$$

since the continuous map

$$N_0 \colon \mathbb{D}\backslash\partial\mathbb{D} \to A \cup B \tag{4.19}$$

has to map the connected domain into a connected component, also compare [1, Cor. IV.19.7]. Since the height function

$$w = \langle X, e_0 \rangle \tag{4.20}$$

is increasing at the boundary, cf. Lemma 3.7, it attains an interior minimum and thus $-e_0 \in N_0(\mathbb{D}\backslash\partial\mathbb{D})$. Thus we must have (4.17). A closed geodesic ball $\mathcal{B}$ around e_0 satisfying

$$e_0 \in \mathcal{B} \subset \operatorname{conv}(\partial M_0) \tag{4.21}$$

satisfies the following inclusion for the polar convex bodies,

$$\bar{A} = \operatorname{conv}(\partial M_0)^* \subset \mathcal{B}^*, \tag{4.22}$$

compare [6, Cor. 9.2.10] for the inclusion and [6, eq. (9.2.65)] for the equality. This implies the claim, since $\mathcal{B}^*$ is a geodesic ball around $-e_0$, the size of which can be estimated from above in dependence of the size of $\mathcal{B}$, meaning that all normals to ∂M_0 are strictly pointing downwards. □ □

4.5. **Corollary.** *In the situation of Lemma 4.4 the height function*

$$w = \langle X_0, e_0 \rangle \tag{4.23}$$

does not attain an interior local maximum.

Proof. Using the Gaussian formula we obtain

$$\Delta w = -H\varphi > 0. \tag{4.24}$$

□ □

4.6. **Corollary.** *In the situation of Lemma 4.4 there holds*

$$\langle e_0 - X_0, N_0 \rangle < 0. \tag{4.25}$$

Proof. Suppose the claim to be false, then there existed a point $z \in \operatorname{int}(\mathbb{D})$ with the property that e_0 is not contained in the supporting open halfspace at $X_0 = X_0(z)$,

$$S_0 = \{x \in \mathbb{R}^{n+1} : \langle x - X_0, N_0 \rangle < 0\}. \tag{4.26}$$

Due to Lemma 4.4 we then also had

$$0 \notin \bar{S}_0. \tag{4.27}$$

By the strict convexity of M_0 we have

$$X_0(\partial\mathbb{D}) \subset S_0. \tag{4.28}$$

∂S_0 splits $\mathbb{S}^n$ into two spherical caps. Translating the hyperplane ∂S_0 until it hits 0, we see that ∂M_0 originally had to be contained in the spherical cap which is geodesically convex. But by assumption we have $e_0 \in \operatorname{int}(\operatorname{conv}(\partial M_0))$, which contradicts $e_0 \notin S_0$. □ □

We are now able to estimate the height of a hypersurface M_0 as the latter appears in (1.4). It depends on the estimate in Lemma 4.4 and the curvature.

4.7. **Lemma.** *In the situation of Lemma 4.4 the height*

$$w = \langle X_0, e_0 \rangle \tag{4.29}$$

satisfies

$$w \geq \delta > 0, \tag{4.30}$$

for a constant δ, which depends on the constant c_0 in Lemma 4.4, the length of the second fundamental form of M_0 and the distance of ∂M_0 to the equator $\mathcal{S}(e_0)$.

Proof. Let $a \in M_0$ be the interior global minimum point of w. Due to Lemma 4.4 it is possible to write M_0 locally around a as a graph over the unit disk in $\{0\} \times \mathbb{R}^n$, where w is the graph function. Then

$$w_{ij} = -h_{ij}\langle N_0, e_0 \rangle. \tag{4.31}$$

Using [6, Lemma 2.7.6], we obtain that the Hessian of w with respect to Euclidean coordinates only depends on the second fundamental form and on the estimate of $\langle N, e_0 \rangle$ from below. Define

$$\hat{M}_0 = \bigcap_{y \in M_0} \{x \in \mathbb{R}^{n+1} : \langle x - y, N_0 \rangle \leq 0\}. \tag{4.32}$$

From the previous considerations $\hat{M}_0$ satisfies an interior sphere condition at a with interior ball B_R depending on $\sup \|A\|$ and $\langle N, e_0 \rangle$. Due to

$$B_R \subset \hat{M}_0 \subset C_0, \tag{4.33}$$

from Lemma 4.3 we obtain the existence of $\delta > 0$, such that

$$\langle a, e_0 \rangle \geq \delta. \tag{4.34}$$

□ □

4.8. **Corollary.** *In the situation of Lemma 4.4 we have*

$$X_0(\operatorname{int}(\mathbb{D})) \subset \operatorname{int}(B^+), \tag{4.35}$$

where $B^+ \subset \mathbb{R}^{n+1}$ is the pointed halfball

$$B^+ = B_1^+(0) \backslash \{e_0\}. \tag{4.36}$$

Proof. The function

$$\rho = |X_0|^2 \tag{4.37}$$

satisfies

$$\Delta\rho = -2H\langle N_0, X_0\rangle + 2n, \tag{4.38}$$

due to the Gaussian formula. At an interior maximum of ρ we have

$$0 = \nabla\rho \tag{4.39}$$

and thus X_0 has to be a multiple of N_0. Since

$$\langle X_0, e_0\rangle > 0 \tag{4.40}$$

due to Lemma 4.7 and

$$\langle N_0, e_0\rangle < 0 \tag{4.41}$$

due to Lemma 4.4, we have

$$\langle N_0, X_0\rangle < 0. \tag{4.42}$$

Thus at a maximal point we have

$$\Delta\rho > 0, \tag{4.43}$$

a contradiction. Since we have $\rho = 1$ at the boundary, the claim follows. □ □

5. Moebius coordinates and the scalar flow

In this section we want to derive a scalar flow equation naturally associated with (1.2). Therefore we aim for a graph representation. A natural candidate for hypersurfaces of our type are rotations of Moebius transformations on the plane. Consider a one-parameter family of Moebius transformations of the form

$$\tilde{f}(x,\lambda) = \frac{(1+\lambda)x + i(\lambda-1)}{1+\lambda+i(1-\lambda)x}, \tag{5.1}$$

where $(x,\lambda) \in [-1,1]\times[1,\infty)$. For each λ this is a conformal transformation moving the real axis towards i as $\lambda\to\infty$, whereas the boundary of the real interval $[-1,1]$ maps to the unit sphere perpendicularly. A rotation of a plane in $\mathbb{R}^{n+1}$ around the e_0-axis gives rise to the following definition.

5.1. **Definition.** Let $D \subset \mathbb{R}^n$ be the unit disk. Define *Moebius coordinates* for the pointed halfball

$$B^+ := B_1^+(0)\backslash\{e_0\} \tag{5.2}$$

to be the diffeomorphism

$$\begin{aligned} f\colon D\times[1,\infty) &\to B^+ \\ f(x,\lambda) &= \frac{4\lambda x + (1+|x|^2)(\lambda^2-1)e_0}{(1+\lambda)^2 + (1-\lambda)^2|x|^2}. \end{aligned} \tag{5.3}$$

Graphs in Moebius coordinates. Let us provide some general formulae for hypersurfaces $M \subset \mathbb{R}^{n+1}$ which can be written as graphs in Moebius coordinates. Thus suppose the embedding of a hypersurface M is given by a map

$$\begin{aligned} X\colon \mathbb{D} &\hookrightarrow \mathbb{R}^{n+1} \\ z &\mapsto f(x(z), u(x(z))), \end{aligned} \tag{5.4}$$

where $u\colon D \to [1, \infty)$ is a function. First of all, from a tedious computation and the conformality of f we obtain a representation of the Euclidean metric $\delta_{\alpha\beta}$ in Moebius coordinates,

$$d\bar{s}^2 = e^{2\psi}({dx^0}^2 + \sigma_{ij}dx^i dx^j), \tag{5.5}$$

where x^0 corresponds to the λ-coordinate,

$$e^{2\psi} = \left\langle \frac{\partial f}{\partial x^0}, \frac{\partial f}{\partial x^0} \right\rangle, \tag{5.6}$$

$$\frac{\partial f}{\partial \lambda}(x, \lambda) = \frac{(1 + |x|^2)(1 - \lambda^2)}{\lambda((1 + \lambda)^2 + (1 - \lambda)^2|x|^2)} \left(f - \frac{\lambda^2 + 1}{\lambda^2 - 1} e_0 \right). \tag{5.7}$$

and

$$\sigma_{ij} = e^{-2\psi} \left\langle \frac{\partial f}{\partial x^i}, \frac{\partial f}{\partial x^j} \right\rangle. \tag{5.8}$$

For M we have the induced metric

$$g_{ij} = e^{2\psi}(u_i u_j + \sigma_{ij}) \tag{5.9}$$

with inverse

$$g^{ij} = e^{-2\psi} \left(\sigma^{ij} - \frac{\sigma^{ik}u_k}{v} \frac{\sigma^{lj}u_l}{v} \right), \tag{5.10}$$

where

$$v^2 = 1 + \sigma^{ij}u_i u_j. \tag{5.11}$$

The contravariant version of the normal is

$$(N^\alpha) = \pm v^{-1} e^{-\psi}(1, -\sigma^{ik}u_k). \tag{5.12}$$

Those formulae can be found in [6, Sec. 1.5].

Due to the conformality of f the outward Euclidean unit normal to D, $\check{N}$, is mapped to a multiple of the unit normal to the sphere in $\mathbb{R}^{n+1}$ which we called $\tilde{N}$ earlier. Thus for a hypersurface satisfying the boundary condition (1.4c) we obtain

$$\begin{aligned} 0 &= \left\langle \check{N}^k \frac{\partial f}{\partial x^k}, N \right\rangle \\ &= \mp \frac{e^\psi}{v} \check{N}^k u_k \end{aligned} \tag{5.13}$$

and thus such a hypersurface satisfies the Neumann boundary condition

$$\check{N}^k u_k = 0. \tag{5.14}$$

Now we prove that hypersurfaces satisfying (1.4) are graphs in Moebius coordinates.

5.2. **Proposition.** *Let*

$$X_0 \colon M \hookrightarrow \mathbb{R}^{n+1} \tag{5.15}$$

be the embedding of a strictly convex hypersurface M_0, such that (1.4) *holds. Choose $e_0 \in$ $\operatorname{int}(\operatorname{conv}(\partial M_0))$, such that $\operatorname{conv}(\partial M_0)$ is contained in the open hemisphere $\operatorname{int}(\mathcal{H}(e_0))$. Then M_0 can be written as a graph in Moebius coordinates around e_0, i.e. Moebius coordinates in the pointed half-ball $B_1^+(0)\backslash\{e_0\}$ yield a representation*

$$X_0(z) = f(x, u_0(x)), \tag{5.16}$$

where f is the diffeomorphism defined in (5.3).

Proof. Due to Corollary 4.8 Moebius coordinates are well-defined throughout M_0. By the implicit function theorem all we have to show is that

$$\left\langle \frac{\partial f}{\partial \lambda}, N_0 \right\rangle < 0. \tag{5.17}$$

Due to Lemma 4.7 we have $\lambda \geq c > 1$ and thus it suffices to discard the negative scalar fraction in (5.7). We have

$$\begin{aligned} \left\langle X_0 - \frac{\lambda^2+1}{\lambda^2-1} e_0, N_0 \right\rangle &= \langle X_0 - e_0, N_0 \rangle - \left\langle \frac{2}{\lambda^2-1} e_0, N_0 \right\rangle \\ &> -\frac{2}{\lambda^2-1} \langle e_0, N_0 \rangle \\ &> 0, \end{aligned} \tag{5.18}$$

due to Lemma 4.4 and Corollary 4.6. □ □

The previous considerations allow us to naturally associate a scalar parabolic equation to strictly convex solutions of our inverse mean curvature flow (1.2).

5.3. **Corollary.** *Let X be a solution of* (1.2) *on a time interval $[0, \epsilon)$, such that all M_t, $0 \leq t < \epsilon$, range within a pointed halfball B^+ and are graphs in Moebius coordinates for B^+,*

$$M_t = \{(x(t, z), u(t, x)) \colon (t, z) \in [0, \epsilon) \times \mathbb{D}\}. \tag{5.19}$$

Then u solves a parabolic Neumann problem on $[0, \epsilon) \times D$, namely

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{v}{e^{\psi} H} \text{ in } (0, \epsilon) \times D, \\ u_k \breve{N}^k &= 0 \text{ on } [0, \epsilon) \times \partial D, \\ u &= u_0 \text{ on } \{0\} \times D. \end{aligned} \tag{5.20}$$

Proof. For curvature flows in ambient spaces covered by Gaussian coordinate systems the interior equations are deduced in [6, p. 98-99]. Just note that in our case the normal N_0 and the vector $\frac{\partial f}{\partial x^0}$ are pointing in opposite directions, hence the sign. The boundary equation follows from the fact that all M_t are perpendicular to the sphere and by the derivation of (5.14). □ □

6. Curvature estimates and convexity

6.1. *Remark.* Let T^* be the largest time, such that there exists a smooth solution to (1.2) on the interval $[0, T^*)$. This implies mean convexity of M_t, $0 \leq t < T^*$. By Remark 3.1 we indeed have $T^* > 0$. Let $\bar{T} > 0$ be the largest time, such that the solution is smooth on $[0, \bar{T})$ and M_t is strictly convex for all $0 \leq t < \bar{T}$.

6.2. **Proposition.** *Let X be the solution of (1.2) on the interval $[0, \bar{T})$. Then the principal curvatures are bounded, i.e. for $1 \leq i \leq n$ there holds*

$$\kappa_i \leq H \leq \max_{\mathbb{D}} H(0, \cdot) \quad \forall t \in [0, \bar{T}). \tag{6.1}$$

Proof. Using the convexity of the flow hypersurfaces up to $\bar{T}$, all we have to bound is H. From Lemma 3.8 we obtain

$$\dot{H} - \frac{1}{H^2}\Delta H \leq -\frac{\|A\|^2}{H^2} H \tag{6.2}$$

and

$$H_k \tilde{n}^k = -H. \tag{6.3}$$

Thus the claim follows from a standard maximum principle, e.g. [19, Thm. 3.1]. □ □

6.3. **Lemma.** *On the interval $[0, \bar{T})$ let*

$$y_t \colon \partial\mathbb{D} \hookrightarrow \mathbb{S}^n \tag{6.4}$$

be the induced embeddings of X_t. Then the convex bodies of the embedded submanifolds $\partial M_t \hookrightarrow \mathbb{S}^n$ form an increasing sequence and satisfy uniform interior sphere conditions independently of t.

Proof. The convexity of the ∂M_t in $\mathbb{S}^n$ follow from Lemma 3.6. From (3.8) we see that the enclosed convex bodies are increasing. From Proposition 6.2 and Lemma 3.6 we obtain uniform C^2-estimates and thus uniform interior sphere conditions, also compare [14, Def. 3.2]. □ □

6.4. **Corollary.** *There exists a $C^{1,\alpha}$ limiting surface $\partial M_{\bar{T}}$ arising as the limit of the ∂M_t. $\partial M_{\bar{T}}$ either is an equator of the sphere or is contained in an open hemisphere.*

Proof. $\partial M_{\bar{T}}$ is the boundary of a weakly convex body in a hemisphere, in the sense of [14, Def. 3.2], also compare [14, Lemma 6.1]. [14, Thm. 1.1] implies the claim. □ □

We want to conclude that $\bar{T} = T^*$ and that ∂M_{T^*} must be an equator, which would yield the result due to the height estimates. Therefore we need some more estimates.

6.5. **Lemma.** *Let X be the solution of (1.2) on the interval $[0, \bar{T})$ and suppose that $\partial M_{\bar{T}}$ is not an equator. Then there holds*

$$\sup_{[0,\bar{T})\times\mathbb{D}} \frac{1}{H} \leq c, \tag{6.5}$$

where c depends on M_0 and the distance of $\partial M_{\bar{T}}$ to a suitable equator $\mathcal{S}(e_0)$.

Proof. Let $e_0 \in \operatorname{int}(\operatorname{conv}(\partial M_{\bar T}))$, such that $\operatorname{conv}(\partial M_{\bar T})$ is contained in $\operatorname{int}(\mathcal{H}(e_0))$. Then, due to the monotonicity of $\operatorname{conv}(\partial M_t)$ we also have

$$e_0 \in \operatorname{int}(\operatorname{conv}(\partial M_t)) \tag{6.6}$$

for t close to $\bar T$. Thus it is possible to apply Lemma 4.7 to obtain a positive lower bound for the height function

$$w = \langle X_t, e_0\rangle \geq \delta > 0. \tag{6.7}$$

Define the strictly convex function in $\mathbb{R}^{n+1}$

$$\chi(x) = \frac{1}{2}|\hat{x}|^2 + \frac{\beta}{2}(x^0)^2 - \lambda x^0 + 1, \tag{6.8}$$

where

$$\hat{x} = (0, x^1, \ldots, x^n) \tag{6.9}$$

and

$$\lambda > \frac{1}{\delta}, \quad 0 < \beta < 1. \tag{6.10}$$

Define

$$\zeta = \frac{1}{H}\frac{1}{\frac{1}{2} - \chi} \equiv \frac{1}{H}G(\chi). \tag{6.11}$$

Due to the height estimates, ζ is well defined and positive on $[0, \bar T) \times \mathbb{D}$. With the help of Lemma 3.8 and Lemma 3.10 a simple computation yields the following evolution equation for ζ, namely

$$\begin{aligned} \dot\zeta - \frac{1}{H^2}\Delta\zeta = {} & \frac{\|A\|^2}{H^2}\zeta + 2\chi_\alpha N^\alpha\zeta^2 - \frac{1}{H}\chi_{\alpha\beta}X_i^\alpha X_j^\beta g^{ij}\zeta^2 \\ & - 2\chi_i\chi^i\zeta^3 - \frac{2}{H^2}\left(\frac{1}{H}\right)_i G^i \end{aligned} \tag{6.12}$$

and the boundary equation

$$\zeta_i\tilde{n}^i = \left(1 + G\chi_\alpha\tilde{N}^\alpha\right)\zeta. \tag{6.13}$$

Due to $X = \tilde{N}$ on the boundary, we obtain

$$\chi_\alpha\tilde{N}^\alpha = 1 + (\beta - 1)(X^0)^2 - \lambda X^0 \tag{6.14}$$

and thus on the boundary

$$1 + G\chi_\alpha\tilde{N}^\alpha = 1 + \frac{1 + (\beta - 1)(X^0)^2 - \lambda X^0}{\lambda X^0 - \frac{\beta - 1}{2}(X^0)^2 - 1} < 0. \tag{6.15}$$

Now suppose for $0 < T < \bar T$ that

$$\max_{[0,T]\times\mathbb{D}} \zeta = \zeta(t_0, z_0) \geq 1, \quad t_0 > 0. \tag{6.16}$$

Then $z_0 \in \operatorname{int}(\mathbb{D})$ and thus from (6.12) we obtain at this point that, also using

$$\frac{G_i}{G} = -\frac{\left(\frac{1}{H}\right)_i}{\frac{1}{H}}, \tag{6.17}$$

$$0 \leq \left(c - \frac{1}{H}\chi_{\alpha\beta}X_i^\alpha X_j^\beta g^{ij}\right)\zeta^2, \tag{6.18}$$

where $c = c(\delta)$. Since

$$\chi_{\alpha\beta} X_i^\alpha X_j^\beta g^{ij} = \chi_{\alpha\beta}\bar{g}^{\alpha\beta} - \chi_{\alpha\beta} N^\alpha N^\beta \\ = n + \beta - 1 + (1-\beta)(N^0)^2, \tag{6.19}$$

we obtain a bound for $\frac{1}{H}$ at the point (t_0, z_0). Since G is bounded, this implies a uniform bound on ζ and in turn a uniform bound on $\frac{1}{H}$. □ □

6.6. **Proposition.** *There holds $\bar{T} = T^*$. In particular the strict convexity of the flow hypersurfaces is preserved up to T^*.*

Proof. Suppose that $\bar{T} < T^* \le \infty$. In case that $\partial M_{\bar{T}}$ is an equator of the sphere, we conclude from the height estimates that $M_{\bar{T}}$ is a flat disk and thus a singularity of the flow. This would yield $\bar{T} = T^*$. Thus suppose that $\partial M_{\bar{T}}$ is not an equator. From Lemma 6.5 we obtain

$$\frac{1}{H} \le c \quad \forall t \in [0, \bar{T}) \tag{6.20}$$

and again the height function satisfies

$$w \ge \delta > 0. \tag{6.21}$$

Define

$$\tilde{H} = \sum_{i=1}^n \frac{1}{\kappa_i} = g_{ij}\tilde{h}^{ij}, \tag{6.22}$$

where $(\tilde{h}^{ij})$ is the inverse of (h_{ij}). At a given point choose coordinates with respect to the basis $\mathcal{B} = (\tilde{n}, z_I)$, then at the boundary we deduce, due to Lemma 3.4 and Lemma 3.5, that

$$\begin{aligned} \tilde{H}_k \tilde{n}^k &= -\tilde{h}_i^r \tilde{h}^{si} h_{rs;k}\tilde{n}^k \\ &= -\tilde{h}_1^1 \tilde{h}^{11} h_{11;k}\tilde{n}^k - \tilde{h}_I^J \tilde{h}^{KI} h_{JK;k}\tilde{n}^k \\ &= n\tilde{h}_1^1 \tilde{h}^{11} h_{11} + \tilde{h}_I^J \tilde{h}^{KI} h_{JK} - \tilde{h}_I^J \tilde{h}^{KI} g_{KJ} h_{11} \\ &\le (n-1)\tilde{h}_1^1 + \tilde{h}_i^r \tilde{h}^{si} h_{rs} \\ &= (n-1)\tilde{h}_j^i \tilde{n}_i \tilde{n}^j + \tilde{H}. \end{aligned} \tag{6.23}$$

Set

$$\phi = \log \tilde{H} - (n+1)\log w - \alpha t, \quad t < \bar{T}, \tag{6.24}$$

where α will be chosen in dependence of δ and the initial data. From [5, Lemma 6.5] and Lemma 3.9 we obtain

$$\begin{aligned} \dot{\phi} - \frac{1}{H^2}\Delta\phi = &-\frac{\|A\|^2}{H^2} + \frac{2n}{H\tilde{H}} + \frac{2}{H^2\tilde{H}^2}\tilde{H}_i\tilde{H}^i \\ &- \left(\frac{2}{H^2} g^{rs}\tilde{h}^{kl} h_{rk;p} h_{sl;q} - \frac{2}{H^3} H_p H_q\right)\frac{\tilde{h}^{pi}\tilde{h}_i^q}{\tilde{H}} \\ &- \frac{2n+2}{Hw}\langle N, e_0\rangle - \frac{n+1}{H^2 w^2} w^i w_i - \alpha \end{aligned} \tag{6.25}$$

in the interior and

$$\phi_k \tilde{n}^k \le 1 + \frac{n-1}{\tilde{H}}\tilde{h}_1^1 - (n+1) < -1 \quad \forall (t,\xi) \in [0, \bar{T}) \times \partial\mathbb{D}. \tag{6.26}$$

Now suppose that for $0 < T < \bar{T}$ we have

$$\sup_{[0,T]\times\mathbb{D}} \phi = \phi(t_0, z_0), \quad t_0 > 0. \tag{6.27}$$

Then z_0 does not lie on $\partial\mathbb{D}$. From (6.25) we obtain at (t_0, z_0), also using

$$\frac{\tilde{H}_i}{\tilde{H}} = (n+1)\frac{w_i}{w} \tag{6.28}$$

and that the big bracket is nonnegative by [5, equ. (1.7)], that

$$0 \le c + \frac{2(n+1)^2}{H^2 w^2}\|Dw\|^2 - \alpha, \tag{6.29}$$

where the constant depends on δ and the bound on H^{-1}. For large α this is a contradiction. Thus under the assumption that $\partial M_{\bar{T}}$ is not an equator we obtain that the supremum of ϕ would be decreasing and thus ϕ was bounded up to $\bar{T}$. But then

$$\log \tilde{H} = \phi + (n+1)\log w + \alpha t \le c + \alpha\bar{T}, \tag{6.30}$$

which contradicts the definition of $\bar{T}$, at which $\tilde{H}$ would have to blow up, provided $\bar{T} < T^*$. □ □

6.7. **Corollary.** *There holds*

$$T^* < \infty. \tag{6.31}$$

Proof. Let $e_0 \in \operatorname{int}(\operatorname{conv}(\partial M_{T^*}))$, such that $\operatorname{conv}(\partial M_{T^*}) \subset \mathcal{H}(e_0)$. The induced strictly convex hypersurfaces $\partial M_t \hookrightarrow \mathbb{S}^n$ satisfy the flow equation (3.8), which has a uniformly positive speed in normal direction. Thus ∂M_{T^*} is reached in finite time. □ □

7. Convergence to a flat disk

We have seen that as long as the boundary of the flow is strictly contained in an open hemisphere, we have uniform bounds on the height, the mean curvature and the principal curvatures. We want to conclude that the flow can be extended whenever ∂M_{T^*} is not an equator. This would finish the proof of the main result due to the definition of T^*. In this section we will apply regularity theory to the scalar flow equation in Corollary 5.3 to achieve this.

A straightforward computation yields the following representation of this equation.

7.1. **Proposition.** *The function $u\colon (0,T^*)\times D \to [1,\infty)$ satisfies the equation*

$$\frac{\partial u}{\partial t} = -\frac{v}{e^{2\psi}v^{-1}g^{ij}u_{i,j} + A(x,u,Du)} \equiv F(x,u,Du,D^2u), \tag{7.1}$$

where A is smooth and F is a uniformly parabolic operator, provided ∂M_{T^*} is not an equator of the sphere.

Proof. An easy computation gives a relation between covariant and partial derivatives of u, namely

$$u_{ij} = u_{i,j}v^{-2} + r_{ij}(x,u,Du), \tag{7.2}$$

where r_{ij} is a smooth tensor of the indicated variables. Due to [6, equ. (1.5.10)] we obtain

$$h_{ij}v^{-1}e^{-\psi} = u_{i,j}v^{-2} + r_{ij}(x,u,Du) \tag{7.3}$$

with a possibly different, but still smooth, tensor r_{ij}. Inserting this into (5.20) gives the first equality.

The parabolicity follows from

$$\frac{\partial F}{\partial u_{i,j}} = \frac{v}{e^{\psi} H^2} \frac{\partial H}{\partial u_{i,j}} = \frac{1}{H^2} g^{ij}, \tag{7.4}$$

since as long as ∂M_{T^*} is not an equator, we have $H \geq c > 0$ by Lemma 6.5 and g^{ij} is equivalent to the Euclidean metric on D due to (5.18). □ □

7.2. **Lemma.** *Let $X\colon (0,T] \to \mathbb{R}^{n+1}$ be a solution of* (1.2) *and suppose that ∂M_T is not an equator of the sphere. Then*

$$T^* > T + \epsilon, \tag{7.5}$$

where ϵ depends on M_0 and the distance of ∂M_T to a suitable equator.

Proof. (i) Considering the scalar problem as in Corollary 5.3, from Proposition 7.1 and standard regularity theory we obtain C^∞-estimates up to T for u, compare for example [13, Thm. 14.23] or [20, Thm. 4, Thm. 5]. A slight adjustment of the proof of [6, Thm. 2.5.7] to the Neumann boundary case yields a short-time existence interval of length ϵ for $C^{2,\alpha}$ initial functions, depending on the data of the differential operator. In our situation, these data are uniformly under control, such that choosing a flow hypersurface M_{t_0} with $T - t_0 < \epsilon$ yields an extension beyond T. By the standard method of difference quotients this extension is smooth. Thus we have extended the scalar function u.

(ii) To obtain the full curvature flow from the scalar function u, we use the standard method applied in [15, Sec. 2.3], solving an ODE to allow for normal directed evolution. □ □

Together with Corollary 6.7 and the C^2-estimates we obtain the final result.

7.3. **Corollary.** *∂M_{T^*} is an equator of the sphere and M_{T^*} is an embedded flat disk.*

7.4. *Remark.* From Proposition 6.2 and (7.3) we obtain uniform C^2-bounds for the graph functions u and thus the norm of convergence, in which the flow hypersurfaces converge to unit disk can be characterised by saying that the functions u converge to the constant function with value 1 in the norm of $C^{1,\beta}(D)$.

Appendix

We use this appendix to observe that in general the inverse mean curvature flow with a Neumann boundary condition may *not* be expected to converge globally to a minimal hypersurface as proved above. Indeed, we shall construct a counterexample for boundary manifolds arbitrarily close to the sphere.

We choose Σ be a rotationally symmetric ellipsoid. We will consider equations (1.2) replacing $\mathbb{S}^n$ with Σ and $\tilde{N}$ with the outward normal to Σ. We start from rotationally symmetric, strictly convex initial data and flow by inverse mean curvature flow.

We firstly observe that the flow from such initial data may only exist for a finite time: Suppose not. An easy extension of Lemma 3.3 yields

$$H_i \tilde{n}^i = -H \check{h}_{ij} \nu^i \nu^j,$$

where $\check{h}_{ij}$ is the second fundamental form of the ellipsoid (with respect to the outwards pointing normal vector). Due to the convexity of the ellipsoid, we may proceed as in Proposition 6.2 to get an upper bound on H, and so a lower bound on the speed of the flow. Since the boundary is constrained to move with a speed uniformly bounded below around the ellipsoid, the boundary must meet itself in finite time. At this point from standard calculations on rotationally symmetric surfaces, one principal curvature of the manifold must become infinite. By mean convexity we have a singularity of the flow in the sense that $\|A\|^2$ blows up everywhere on the boundary, and the classical flow must stop. We therefore have that for all such initial data a finite time singularity occurs.

We know that if the flow does converge to a minimal surface then, since rotational symmetry is preserved by the flow, it must converge to either a catenoid or a flat plane. The former of these options is not possible since it necessitates a change of topology, and so due to the boundary condition such a global singularity may only occur at the plane of reflectional symmetry of the boundary ellipsoid.

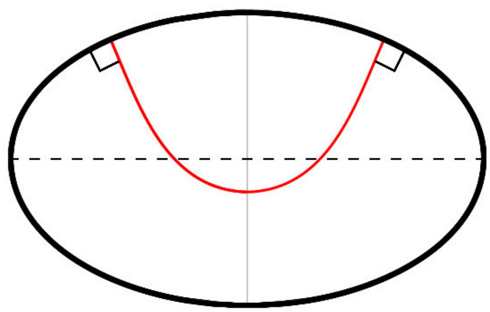

Figure 1. An ellipsoid with strictly convex, rotationally symmetric initial data (red) that cannot converge to the only minimal surface allowed by the boundary condition (dotted).

We may now construct strictly convex, rotationally symmetric initial data as in Figure 1 such that this initial data passes through the minimal surface – such data may always be constructed if the boundary ellipsoid is flattened in the axis of rotation. While the flow remains parabolic it may only move in one direction, and so it can never converge to this plane.

Hence, for any non-spherical rotational ellipsoid, there exist convex rotationally symmetric initial data such the the flow forms a finite time singularity and cannot converge to a (smooth) minimal surface.

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Appendix A7

A GEOMETRIC INEQUALITY FOR CONVEX FREE BOUNDARY HYPERSURFACES IN THE UNIT BALL

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A GEOMETRIC INEQUALITY FOR CONVEX FREE BOUNDARY HYPERSURFACES IN THE UNIT BALL

BEN LAMBERT AND JULIAN SCHEUER

ABSTRACT. We use the inverse mean curvature flow with a free boundary perpendicular to the sphere to prove a geometric inequality involving the Willmore energy for convex hypersurfaces of dimension $n \geq 3$ with boundary on the sphere.

1. Introduction

In [16] we considered the inverse mean curvature flow (IMCF) perpendicular to the sphere, namely a family of embeddings

$$X\colon \mathbb{D} \times [0, T^*) \to \mathbb{R}^{n+1}, \tag{1.1}$$

where $\mathbb{D}$ denotes the n-dimensional unit disk, which satisfy the Neumann boundary value problem

$$\dot{X} = \frac{1}{H} N, \tag{1.2a}$$

$$X(\partial\mathbb{D}) = \partial X(\mathbb{D}) \subset \mathbb{S}^n, \tag{1.2b}$$

$$0 = \left\langle N_{|\partial\mathbb{D}}, \tilde{N}(X_{|\partial\mathbb{D}}) \right\rangle, \tag{1.2c}$$

$$\left\langle \dot{\gamma}(0), \tilde{N} \right\rangle \geq 0 \quad \forall \gamma \in C^1((-\epsilon, 0], M_t)\colon \gamma(0) \in \partial X(\mathbb{D}) \tag{1.2d}$$

with initial embedding X_0 of a strictly convex hypersurface M_0, also satisfying the conditions (1.2b), (1.2c) and (1.2d). Here $\tilde{N}$ denotes the outward unit normal of $\mathbb{S}^n$. In the following we will refer to these three conditions by saying that M_0 is *perpendicular to the sphere from the inside.*

In [16, Thm. 1] we proved that (1.2) with strictly convex initial data exists smoothly up to a maximal time T^*, preserves the strict convexity as well as the perpendicularity condition up to T^* and that T^* is characterised by the $C^{1,\alpha}$-convergence of the embeddings $X(t, \cdot)$ to the embedding of a flat disk bisecting the unit ball, where $\alpha < 1$ is arbitrary; also compare [16, Rem. 1]. Note that the proof of this convergence result heavily depends on the assumption of strict convexity for the initial embedding X_0. This is due to the fact that we obtained the final flat limiting shape at time T^* by applying a rigidity result for weakly convex bodies in the sphere $\mathbb{S}^n$, which was deduced in [19]. In order to arrive at a situation where this rigidity result holds, we needed the strict convexity of X_0. We are not aware of a proof which avoids this assumption and in fact it is an interesting open problem to obtain convergence results for the IMCF perpendicular to the sphere under the assumption of initial mean-convexity, rather than strict convexity.

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However, the object of this paper is different, namely to apply this convergence result to prove a Li-Yau type inequality (cf. [18]) for convex hypersurfaces with boundary in any dimension $n \geq 3$.

1.1. **Theorem.** *Let $n \geq 3$ and $M^n \subset \mathbb{R}^{n+1}$ be a smoothly embedded n-disk, such that M^n is a convex hypersurface perpendicular to $\mathbb{S}^n$ from the inside. Then there holds*

$$\frac{1}{2}|M|^{\frac{2-n}{n}} \int_M H^2 + \omega_n^{\frac{2-n}{n}} |\partial M| \geq \omega_n^{\frac{2-n}{n}} |\mathbb{S}^{n-1}| \tag{1.3}$$

and equality holds if and only if M is a perpendicularly intersecting hyperplane.

Here $|\cdot|$ denotes the respective surface measures of M, ∂M and $\mathbb{S}^{n-1}$ as inherited from $\mathbb{R}^{n+1}$ and ω_n is the volume of the n-dimensional unit ball. We call a hypersurface M *convex*, if there exists a choice of a unit normal vector field, such that all the principal curvatures at any point are non-negative and *strictly convex*, if they are all positive throughout M. Note that convex or strictly convex hypersurfaces with boundary may be way more complicated than in the boundaryless case. In particular the well known supporting hyperplane property in the boundaryless case is not valid without further assumptions if M has nonempty boundary, compare for example the nice treatment of these issues in [11].

In the case of surfaces, $n = 2$, inequalities similar to (1.3) have attracted a lot of attention. In this situation an even sharper version of (1.3) was shown in broader generality than in the restricted class of convex surfaces, and was even demonstrated in higher codimension. Namely, replacing the leading factor 1/2 in (1.3) by 1/4, Volkmann proved the inequality without the convexity assumption in [24]. In the case of higher dimensions less is known, let us only mention a result by Brendle on minimal surfaces, [1]. We refer to the extensive bibliography in [23] for a broader overview over the topic. To our knowledge, the inequality (1.3) has not previously been treated in the higher dimensional hypersurface case.

Let us discuss the well established method of proof of geometric inequalities as in (1.3) using curvature flows. (1.3) makes a statement about a certain class of hypersurfaces M, here smooth and convex ones. To prove this inequality with the help of a specific curvature flow, three things have to be satisfied: First of all M must be an admissible initial hypersurface for the flow, i.e. one has short-time existence with sufficient regularity up to M. Then one has to show that the functional Q, here the left hand side of (1.3), is monotone during the evolution of the flow. Finally we need a convergence result for the flow to a limiting shape in a sufficiently smooth manner. Then we deduce the desired inequality due to the monotonicity of the functional, which yields

$$Q(M) = Q(0) \geq Q(\text{limiting shape}). \tag{1.4}$$

Using this strategy, several geometric inequalities which might have or have not previously been known for convex hypersurfaces could be generalised to a broader class. For example the well known Minkowski inequality for closed convex surfaces in $\mathbb{R}^3$,

$$\frac{1}{\sqrt{|M|}} \int_M H \geq 4\sqrt{\pi} \tag{1.5}$$

with equality if and only if M is a round sphere, was generalised to closed, starshaped and mean-convex surfaces in [12]. This was possible since the inverse mean curvature flow in $\mathbb{R}^{n+1}$ allows such more general hypersurfaces as initial data and the left hand side of (1.5) is decreasing under this flow and constant if and only if it is a flow of spheres. The relevant convergence result for the IMCF in $\mathbb{R}^{n+1}$ was established independently by Gerhardt in [7] and Urbas in [22]. They show that for such initial hypersurfaces the flow expands to infinity and to a round sphere after rescaling. Due to the scale-invariance of the left hand side of

(1.5) the Minkowski inequality follows. Note that once the convergence result is settled, the proof of the inequality is incredibly easy. The same method, also using other flows than the IMCF, was successfully used to prove various kinds of geometric inequalities such as those of Alexandrov-Fenchel type and inequalities for quermassintegrals of convex bodies. Compare [15] for a related inequality in $\mathbb{R}^{n+1}$, [19] and [25] for Alexandrov-Fenchel inequalities in the sphere, [3], [5], [17] and [25] for related results in the hyperbolic space, as well as [2] and [6] in other Riemannian manifolds. The relevant convergence result for the flow in hyperbolic space was established by Gerhardt in [9]. The IMCF in the sphere was treated by Gerhardt in [10] and with different methods by Makowski and the second author in [19]. The probably most famous result in this direction is the proof of the Riemannian Penrose inequality in [14] by Huisken and Ilmanen, which additionally faced the difficulty of singularity formation under the evolution. They overcame this by using the weak notion of IMCF.

In the proof of our proposed inequality (1.3) we try to adapt this method. However, a thorough look at the statement of the theorem and the flow result reveals that the flow result is not available for the whole class of hypersurfaces for which we want to prove the inequality. Namely the flow result requires strict convexity while the inequality is supposed to hold for convex hypersurfaces. This becomes most obvious when looking at the limiting case: A flat disk is certainly a singularity for IMCF and hence there is now way to start the IMCF from it. This introduces an additional complication. The standard proof only works for strictly convex hypersurfaces, the case of which we will treat in section 2. We will resolve the general issue using approximation by strictly convex hypersurfaces. Here the main technical difficulty is that we need an approximation which preserves the perpendicularity to the sphere at boundary points. Fortunately the mean curvature flow serves as a way out, as we will see in section 3. In section 4 we put everything together for the final proof.

We remark that with an improved result on IMCF perpendicular to the sphere which is valid for more general initial hypersurfaces, (1.3) should also be generalisable away from the convex setting.

2. The case of strictly convex hypersurfaces

In order to prove Theorem 1.1 in the strictly convex case we will use the strategy described in the introduction to show that the left hand side of (1.3) is decreasing under the flow and then use the convergence result for the flow to show that it limits into the right hand side of (1.3). Let us first recollect some essential facts proven in [16] which we need in this section.

2.1. *Remark.* (i) In [16] we proved that the IMCF perpendicular to the sphere drives strictly convex initial hypersurfaces M_0 in finite time T^* to a flat disk in $C^{1,\alpha}$. Hence the boundaries $\partial M_t \subset \mathbb{S}^n$ are driven uniformly to an equator $\mathcal{S}$. Let $\mathcal{H}(e_0)$ be the closed hemisphere with center $e_0 \in \mathbb{S}^n$ that contains all the ∂M_t. Then for t close enough to T^* we have

$$\operatorname{dist}(e_0, \partial M_t) \geq c \geq 0 \tag{2.1}$$

and hence by the result in [16, Lemma 11] we have

$$\langle N, e_0 \rangle \leq c_0 < 0 \tag{2.2}$$

for t sufficiently close to T^*. Here $-N$ denotes the unit normal field with respect to which the flow hypersurfaces are strictly convex.

(ii) The next crucial fact, previously applied to give $C^{1,\alpha}$-convergence in [16], is a bound on the principal curvatures (and hence a C^2-estimate). Due to the convexity this is equivalent to a bound on the mean curvature, which was obtained by a standard maximum principle argument: The interior evolution of the negative speed

$$\Phi = -\frac{1}{H} \tag{2.3}$$

along IMCF is given by

$$\dot{\Phi} - \frac{1}{H^2}\Delta\Phi = \frac{\|A\|^2}{H^2}\Phi, \tag{2.4}$$

where Δ is the Laplace-Beltrami operator of the induced metric and $\|A\|$ is the norm of the second fundamental form, cf. [8, Lemma 2.3.4]. Thus the mean curvature satisfies

$$\dot{H} - \frac{1}{H^2}\Delta H \leq -\frac{\|A\|^2}{H^2}H. \tag{2.5}$$

The boundary derivative in our case is

$$\left\langle \nabla H, \tilde{N} \right\rangle = -H, \tag{2.6}$$

compare [16, Lemma 1]. The parabolic maximum principle yields

$$H \leq \max_{M_0} H \tag{2.7}$$

for all times $t < T^*$.

(iii) We will also need that the height

$$w = \langle X, e_0 \rangle \tag{2.8}$$

satisfies

$$\Delta w = -H \langle N, e_0 \rangle \geq -c_0 H, \tag{2.9}$$

where the equality is due to the Gaussian formula and the inequality holds due to (2.2). At the boundary the height function satisfies

$$\left\langle \nabla w, \tilde{N} \right\rangle = w, \tag{2.10}$$

cf. [16, Lemma 5] for a proof.

(iv) The embeddings $X(t, \cdot)\colon \mathbb{D} \to \mathbb{R}^{n+1}$ restrict to embeddings

$$y_t\colon \partial\mathbb{D} \to \mathbb{S}^n. \tag{2.11}$$

As the embeddings $X(t, \cdot)$ give strictly convex hypersurfaces, the y_t yield strictly convex hypersurfaces of the sphere $\mathbb{S}^n$, cf. [16, Lemma 4] for the simple proof. Since the flow of $X(t, \cdot)$ is smooth up to the boundary by standard regularity theory, the y_t themselves satisfy a curvature flow equation in the sphere, namely

$$\dot{y} = \frac{1}{H}\nu, \tag{2.12}$$

where H is the full mean curvature of M_t restricted to $\partial\mathbb{D}$ and ν is the pullback of the normal N along the embedding $x\colon \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$, cf. [16, equ. (20)] for a detailed derivation.

We can now prove the monotonicity of the curvature functional. For this purpose we need control on the L^2-norm of H.

2.2. **Lemma.** *Let the family* (M_t) *of strictly convex hypersurfaces evolve by* (1.2). *Then for all* $1 \leq p < \infty$ *there holds*

$$\lim_{t \to T^*} \int_{M_t} H^p(\cdot, t) = 0. \tag{2.13}$$

Proof. Combine (2.2), (2.9) and (2.10) to deduce

$$\int_{M_t} H \leq -c_0^{-1}\int_{M_t}\Delta w = -c_0^{-1}\int_{\partial M_t} w \to 0, \quad t\to T^*, \tag{2.14}$$

where the latter convergence follows since the boundaries $\partial M_t\subset\mathbb{S}^n$ converge to the equator in C^1. The complete result follows due to the boundedness of H, (2.7), and interpolation. □

Now we can prove Theorem 1.1 in the special case of a strictly convex hypersurface, which will also be needed in the proof of the limiting case.

2.3. **Lemma.** *Let $n\geq 2$ and $M\subset\mathbb{R}^{n+1}$ be a smooth and strictly convex hypersurface perpendicular to $\mathbb{S}^n$ from the inside. Then there holds*

$$\frac{1}{2}|M|^{\frac{2-n}{n}}\int_M H^2 + \omega_n^{\frac{2-n}{n}}|\partial M| > \omega_n^{\frac{2-n}{n}}|\mathbb{S}^{n-1}|. \tag{2.15}$$

Proof. Rewriting (2.4) gives

$$\dot H = \Delta\left(-\frac{1}{H}\right) - \frac{\|A\|^2}{H} \tag{2.16}$$

and [8, Lemma 2.3.1] yields the evolution of the volume element

$$\frac{d}{dt}\sqrt{\det(g_{ij})} = \sqrt{\det(g_{ij})}. \tag{2.17}$$

Thus

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_{M_t} H^2 d\mu_t\right) &= \int_{M_t} H\Delta\left(-\frac{1}{H}\right)d\mu_t - \int_{M_t}\|A\|^2 d\mu_t + \frac{1}{2}\int_{M_t} H^2 d\mu_t\\ &= -\int_{M_t}\frac{\|\nabla H\|^2}{H^2}d\mu_t - \int_{M_t}\|A\|^2 d\mu_t + \frac{1}{2}\int_{M_t} H^2 d\mu_t\\ &\quad - |\partial M_t|,\end{aligned} \tag{2.18}$$

where we used the divergence theorem and (2.6). Since

$$\|A\|^2 = \|\mathring{A}\|^2 + \frac{1}{n}H^2, \tag{2.19}$$

we have

$$\frac{1}{2}H^2 - \|A\|^2 = \frac{n-2}{2n}H^2 - \|\mathring{A}\|^2 \tag{2.20}$$

and thus

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_{M_t} H^2 d\mu_t\right) &= -\int_{M_t}\frac{\|\nabla H\|^2}{H^2}d\mu_t - \int_{M_t}\|\mathring{A}\|^2 d\mu_t\\ &\quad + \frac{n-2}{2n}\int_{M_t} H^2 d\mu_t - |\partial M_t|.\end{aligned} \tag{2.21}$$

Furthermore, due to (2.12) the volume elements of the induced hypersurfaces

$$y_t\colon \partial\mathbb{D}\to\mathbb{S}^n \tag{2.22}$$

satisfy

$$\frac{d}{dt}\sqrt{\det(\gamma_{IJ})} = \frac{\gamma^{IJ}\eta_{IJ}}{H}\sqrt{\det(\gamma_{IJ})} < \sqrt{\det(\gamma_{IJ})}, \tag{2.23}$$

where γ_{IJ} and η_{IJ} denotes the metric and the second fundamental form of these hypersurfaces respectively. Define

$$Q(t) = \frac{1}{2}|M_t|^{\frac{2-n}{n}} \int_{M_t} H^2 + \omega_n^{\frac{2-n}{n}} |\partial M_t|. \tag{2.24}$$

By the previous calculations we have

$$\begin{aligned}\dot{Q}(t) &< \frac{2-n}{2n}|M_t|^{\frac{2-n}{n}} \int_{M_t} H^2 + \frac{n-2}{2n}|M_t|^{\frac{2-n}{n}} \int_{M_t} H^2 - |M_t|^{\frac{2-n}{n}}|\partial M_t| \\ &\quad + \omega_n^{\frac{2-n}{n}} |\partial M_t| \\ &= \left(\omega_n^{\frac{2-n}{n}} - |M_t|^{\frac{2-n}{n}}\right) |\partial M_t| \\ &\leq 0,\end{aligned} \tag{2.25}$$

since we already know by [16, Thm. 1] that $|M_t|$ is increasingly converging to ω_n. Furthermore we know by Lemma 2.2 that

$$\int_{M_t} H^2 \to 0 \tag{2.26}$$

and thus we obtain

$$Q(0) > Q(T^*) = \omega_n^{\frac{2-n}{n}} |\mathbb{S}^{n-1}|. \tag{2.27}$$

□

We also need the following exact description of the maximal time of existence of a smooth solution to (1.2).

2.4. **Lemma** (Exact existence time)**.** *Suppose the initial data M_0 to* (1.2) *is strictly convex. Then the maximal time of existence T^* is*

$$T^* = \log\left(\frac{\omega_n}{|M_0|}\right). \tag{2.28}$$

In particular we obtain the volume estimate

$$|M_0| < \omega_n. \tag{2.29}$$

Proof. Using (2.17), we see that $\frac{d}{dt}|M_t| = |M_t|$ and so

$$|M_t| = e^t|M_0|. \tag{2.30}$$

Since we know that the maximal time is when the flow becomes a flat disk and the flow converges in $C^{1,\beta}$, we know $\omega_n = e^{T^*}|M_0|$ and the equation follows. □

3. Approximation of weakly convex hypersurfaces

One of the main difficulties in proving Theorem 1.1 is the lack of information about the IMCF for weakly convex hypersurfaces. The proof of the result in [16] makes essential use of the strict convexity. Hence it is not straightforward to obtain the limiting case in Theorem 1.1. We will use approximation by strictly convex hypersurfaces to overcome this obstacle. To do this we use the mean curvature flow with the same Neumann boundary condition. More specifically, we still assume M_0 is parametrised by $X_0 : \mathbb{D} \to \mathbb{R}^{n+1}$. Contrary to our previous solution X of the inverse mean curvature flow, we now consider the solution $F : \mathbb{D} \times [0, T) \to \mathbb{R}^{n+1}$ of the mean curvature flow with Neumann boundary condition, i.e.

$$\dot{F} = -HN, \tag{3.1a}$$
$$X(\partial\mathbb{D}) = \partial X(\mathbb{D}) \subset \mathbb{S}^n, \tag{3.1b}$$
$$0 = \left\langle N_{|\partial\mathbb{D}}, \tilde{N}(X_{|\partial\mathbb{D}})\right\rangle, \tag{3.1c}$$
$$\left\langle \dot{\gamma}(0), \tilde{N}\right\rangle \geq 0 \quad \forall \gamma \in C^1((-\epsilon, 0], M_t) \colon \gamma(0) \in \partial X(\mathbb{D}) \tag{3.1d}$$

with initial embedding X_0.

Properties of such mean curvature flows with boundary conditions were studied by A. Stahl in [21] and [20]. Now we use Stahl's short time existence result [21, Thm. 2.1] in conjunction with the following strong maximum principle statement to obtain strictly convex approximating hypersurfaces arbitrarily close to M_0 in $C^{2,\alpha}$. First we need a lemma to ensure that a nontrivial M has a strictly convex point.

3.1. **Lemma.** *Let $M \subset \mathbb{R}^{n+1}$ be a smooth and weakly convex hypersurface perpendicular to $\mathbb{S}^n$ from the inside with embedding vector X. Then either ∂M is an equator of the sphere or there exists $x \in \mathbb{D}$ such that the second fundamental form of M at x is positive definite.*

Proof. As mentioned in Remark 2.1, item (iv), $\partial M \subset \mathbb{S}^n$ is a convex hypersurface of the sphere which is either an equator or strictly contained in an open hemisphere by the classical results in [4]. In the first case we are done. In the second case we pick a point $e_0 \in \operatorname{conv}(\partial M) \subset \mathbb{S}^n$, where the latter denotes the spherical convex body bounded by ∂M, such that also

$$\partial M \subset \operatorname{int}(\mathcal{H}(e_0)), \tag{3.2}$$

where $\mathcal{H}(e_0)$ denotes the closed hemisphere with center e_0. By (2.10) the height

$$w = \langle X, e_0\rangle \tag{3.3}$$

over the hyperplane $e_0^{\perp}$ attains its global minimum in the interior of $\mathbb{D}$. By attaching a large supporting sphere to M from below we find the existence of a strictly convex point. □

Now we can prove the approximation result. A similar technique was used in [13].

3.2. **Theorem.** *Suppose $F \colon \mathbb{D} \times [0, T) \to \mathbb{R}^{n+1}$ is a solution to* (3.1) *with initial hypersurface M_0 being weakly convex and perpendicular to the sphere from the inside. Then either ∂M_0 is an equator of the sphere or $(h_{ij}) > 0$ for $t > 0$.*

Proof. If ∂M_0 is not an equator, then due to Lemma 3.1 there exists a strictly convex point.

Let

$$\chi(x,t) = \min_{|V|=1} h_{ij}V^iV^j. \tag{3.4}$$

Due to the smoothness of h_{ij}, $\chi(x,t)$ is Lipschitz continuous in space and therefore by a simple cut-off function argument we find a smooth function $\phi_0 : M^n \to \mathbb{R}$ so that $0 \leq \phi_0 \leq \chi(x, 0)$ and there exists $y \in M^n$ so that $\phi_0(y) > 0$. We now extend this function to $\phi : \mathbb{D}^n \times [0, \delta) \to \mathbb{R}$ by a heat flow,

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right)\phi = 0 & \text{on } \operatorname{int}(\mathbb{D}) \times [0, \tau) \\ \nabla_\mu \phi = 0 & \text{on } \partial\mathbb{D} \times [0, \tau) \\ \phi(\cdot, 0) = \phi_0(\cdot), \end{cases} \tag{3.5}$$

where Δ is the time dependent Laplace-Beltrami operator of the metrics induced by the solution F of (3.1). This is a linear parabolic PDE and so by standard theory a solution exists for a short time $\tau > 0$. By the strong maximum principle (e.g. [21, Cor. 3.2]), for $t > 0$ we have $\phi(\cdot, t) > 0$ in $\mathbb{D}$.

We now consider

$$M_{ij} = h_{ij} - \phi g_{ij} \tag{3.6}$$

as long both the MCF and the heat flow exist, say for $0 \leq t < \tau$. We know that at time $t = 0$ we have $M_{ij} \geq 0$ by construction of ϕ. We now aim to apply the weak maximum principle with Neumann boundary conditions, [21, Thm. 3.3, Lemma 3.4].

Using the evolution equations in [20, p. 432], we have that on the flowing manifold

$$\left(\frac{\partial}{\partial t} - \Delta\right) M_{ij} = |A|^2 h_{ij} - 2H h_i^k h_{kj} + 2\phi H h_{ij} =: N_{ij}. \tag{3.7}$$

We see that for a unit vector v such that

$$M_{ij} v^i = h_{ij} v^i - \phi g_{ij} v^i = 0, \tag{3.8}$$

we obtain

$$N_{ij} v^i v^j = |A|^2 \phi - 2H\phi^2 + 2H\phi^2 = |A|^2 \phi \geq 0, \tag{3.9}$$

that is, the evolution of M_{ij} satisfies a null eigenvector condition.

For a better comparability to the results in [21] and [20] we switch to Stahl's notation, so that for $p \in \mathbb{S}^n$ write $\mu \in T_p M$ for the outward pointing normal to $\mathbb{S}^n$. Due to [20, Thm. 4.3 (i)], at a point $p \in \partial M$ for basis tangent vectors $\partial_I \in T_p M \cap T_p \mathbb{S}$, the basis

$$\mathcal{B} = (\mu, \partial_I)_{2 \leq I \leq n} \tag{3.10}$$

induces the coordinate representation $M_{I\mu} = 0$. That is μ is both an eigenvector of M_{ij} and a principal direction at the boundary. We now demonstrate that the conditions of [21, Lemma 3.4] hold. For $\partial_I, \partial_J \in T_p M \cap T_p \mathbb{S}^n$, [20, Thm. 4.3 (ii), (iii)] give

$$\nabla_\mu M_{IJ} = h_{\mu\mu} \delta_{IJ} - h_{IJ}, \qquad \nabla_\mu M_{\mu\mu} = 2H - n h_{\mu\mu}. \tag{3.11}$$

We suppose first that $V \in T_p(\partial M)$ is a minimal eigenvector with eigenvalue $\lambda \in (-\delta, 0]$, that is

$$M_{ij} V^i = \lambda g_{ij} V^i. \tag{3.12}$$

We see that V is also a minimal eigenvector of h_{ij}, and therefore

$$h_{ij} V^i V^j \leq h_{\mu\mu}. \tag{3.13}$$

Equation (3.11) now implies $\nabla_\mu M_{IJ} V^I V^J \geq 0$.

Now suppose that μ is a minimal eigenvector with eigenvalue $\lambda \in (-\delta, 0]$. Again minimality of μ implies that for all $W \in T_p(\partial M)$ there holds

$$h_{ij} W^i W^j \geq h_{\mu\mu}. \tag{3.14}$$

In particular this implies $H \geq n h_{\mu\mu}$, and so $\nabla_\mu M_{\mu\mu} \geq H \geq 0$, where we used [20, Thm. 3.1].

We may now apply [21, Thm. 3.3, Lemma 3.4], to give that $M_{ij} \geq 0$. Since $\phi > 0$ for $t > 0$, $h_{ij} > 0$ for $\tau > t > 0$. This then holds for all time that the flow exists by applying [20, Prop. 4.5] to the mean curvature flow defined by $F(x, t - \frac{\tau}{2})$. □

3.3. **Corollary.** *Suppose M is a weakly convex hypersurface perpendicular to the sphere from the inside, such that ∂M is not an equator. Then there exists an $\epsilon > 0$ such that for $0 \leq t < \epsilon$*

there are smooth and strictly convex hypersurfaces perpendicular to the sphere from the inside and satisfy

$$\int_{M_t} H^2 \to \int_M H^2, \quad |M_t| \to |M|, \quad |\partial M_t| \to |\partial M|$$

as $t \to 0$.

Proof. By [21, Thm. 2.1] there exists a solution to equation (3.1) for $F \in C^\infty(\mathbb{D} \times (0,\epsilon)) \cap C^{2+\alpha;1+\frac{\alpha}{2}}(\mathbb{D} \times [0,\epsilon))$. The convergence then follows due to the regularity of the flow at $t = 0$. $\square$

Now we can prove a crucial estimate for the volume.

3.4. **Lemma** (Volume estimate)**.** *Let* M *be a weakly convex hypersurface perpendicular to the sphere from the inside such that* ∂M *is not an equator. Then there holds*

$$|M| \le \omega_n - c_{\partial M}, \tag{3.15}$$

where $c_{\partial M} > 0$ *is a constant only depending on the outer radius of* $\partial M \subset \mathbb{S}^n$*, in the sense that it tends to zero only if the outer radius tends to* $\pi/2$.

Proof. ∂M is a convex hypersurface of the sphere. Since it is not an equator, it is strictly contained in an open hemisphere $\mathrm{int}(\mathcal{H}(e_0))$ with $e_0 \in \mathrm{conv}_{\mathbb{S}^n}(\partial M)$ by [4]. Pick a geodesic ball B_R of radius $R < \pi/2$ around e_0 such that

$$\partial M \subset B_R \tag{3.16}$$

and denote $S_R = \partial B_R$. Use Corollary 3.3 to obtain a strictly convex hypersurface $\tilde{M} \subset \mathbb{R}^{n+1}$, such that $\partial\tilde{M} \subset B_R$. From [10] or also [19] we know that the IMCF for strictly convex closed hypersurfaces of the unit sphere converges in finite time to an equator. By the avoidance principle the IMCF starting at $\partial\tilde{M}$ exists longer that the one starting from S_R. The existence time T_R of the latter flow however can be calculated explicitly in terms of $\pi/2 - R$. Since the volume element also grows exponentially along the IMCF in the sphere and limits to the volume of the equator, we must have

$$|\mathbb{S}^{n-1}| - |\partial\tilde{M}| \ge c_R > 0. \tag{3.17}$$

Now start the IMCF perpendicular to the sphere from $\tilde{M}$. Due to (2.23) the boundary measures $|\partial\tilde{M}|$ grow less than exponentially, but they must still limit to $|\mathbb{S}^{n-1}|$ in finite time. Hence the existence time of this flow is uniformly bounded below in terms of R and in turn we must have

$$|\tilde{M}| \le \omega_n - \tilde{c}_R \tag{3.18}$$

with a new positive constant $\tilde{c}_R$ due to the exponential growth of the area measure and the convergence result. Taking $\tilde{M}$ arbitrarily close to M yields the result. $\square$

4. Proof of Theorem 1.1

If ∂M is an equator, (1.3) is trivial. Due to Corollary 3.3 we see that (1.3) now also holds for weakly convex hypersurfaces. So all we have to prove is the characterisation of the limit. So suppose that (1.3) holds with equality. If ∂M is an equator, then M must be a convex minimal surface, hence totally umbilic and hence a hyperplane. So we may suppose that ∂M is not an equator, which in particular implies that

$$|M| \le \omega_n - c_{\partial M}, \tag{4.1}$$

where we used Lemma 3.4.

Due to Corollary 3.3 for every $\epsilon > 0$ there exists a strictly convex hypersurface perpendicular to the sphere from the inside M^ϵ such that

$$Q(M^\epsilon) \leq Q(M) + \epsilon, \tag{4.2}$$

where $Q(M)$ is the quantity in (2.24) evaluated at the hypersurface M. Starting the flow (1.2) with initial hypersurface M_ϵ, flow hypersurfaces M_t^ϵ and maximal existence time

$$T_\epsilon^* = \log\left(\frac{\omega_n}{|M^\epsilon|}\right), \tag{4.3}$$

in view of (2.25) the corresponding quantities $Q^\epsilon(t)$ satisfy

$$\begin{aligned} \dot{Q}^\epsilon(t) &\leq \left(\omega_n^{\frac{2-n}{n}} - |M_t^\epsilon|^{\frac{2-n}{n}}\right)|\partial M_t^\epsilon| \\ &= \omega_n^{\frac{2-n}{n}}\left(1 - e^{\frac{n-2}{n}(T_\epsilon^*-t)}\right)|\partial M_t^\epsilon|. \end{aligned} \tag{4.4}$$

Due to Lemma 3.4 and Corollary 3.3 there exists a positive time T which only depends on $|M|$ and is independent of ϵ, such that

$$T_\epsilon^* \geq 2T > 0. \tag{4.5}$$

Hence for all ϵ and all $0 \leq t \leq T$ there holds

$$\dot{Q}^\epsilon(t) \leq -c\left(1 - e^{\frac{n-2}{n}T}\right) \equiv -c, \tag{4.6}$$

where $c > 0$ only depends on n, $|M|$ and $|\partial M|$. Using the the strict convexity of M_ϵ and Lemma 2.3, we obtain that

$$\begin{aligned} \omega_n^{\frac{2-n}{n}}|\mathbb{S}^{n-1}| < Q^\epsilon(T) &= Q(M_\epsilon) + \int_0^T \dot{Q}^\epsilon(s)\, ds \\ &\leq Q(M) + \epsilon - cT \\ &= \omega_n^{\frac{2-n}{n}}|\mathbb{S}^{n-1}| + \epsilon - cT, \end{aligned} \tag{4.7}$$

giving a contradiction for small ϵ and completing the proof.

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