

Stability for anisotropic curvature functionals

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Happy birthday Michael and Guofang



Scan me for the slides!

Closed Soap Bubbles

Isoperimetric problem: Determine properties of area minimising surface, given volume constraint.

Round spheres in \mathbb{R}^{n+1} are unique closed minimisers of

$$\mathcal{R}(\Omega) = \frac{\text{Area}^{\frac{n+1}{n}}(\partial\Omega)}{\text{Volume}(\Omega)}$$

Standard variational methods:

- ▶ Minimisers of \mathcal{R} have **constant mean curvature**

$$H = \text{tr}(A) = \sum_{i=1}^n \kappa_i$$

(κ_i) are eigenvalues of the Weingarten map A , principal curvatures.

Alexandrov's theorem

Is a closed embedded constant mean curvature (CMC) hypersurface of \mathbb{R}^{n+1} necessarily a sphere?

Answer: **YES!** (Alexandrov¹)

- ▶ Proof: Reflection across moving planes and the maximum principle.
- ▶ We are going to see another elegant proof today.

Relaxed CMC condition: Suppose for some $\delta > 0$, on a hypersurface M

$$n - \delta \leq H \leq n + \delta.$$

Can we conclude

$$\text{dist}(M, S) \leq C\epsilon$$

for the unit sphere S , a constant C and where

$$\lim_{\delta \rightarrow 0} \epsilon(\delta) = 0 ?$$

¹A characteristic property of spheres, Ann. Mat. Pura Appl. **58** (1962), no. 4, 303–315.

The question of stability

Theorem (Giulio Ciraolo and Luigi Vezzoni)

*A sharp quantitative version of Alexandrov's theorem via the method of moving planes, J. Eur. Math. Soc. **20** (2018), no. 2, 261–299.*

Let Ω be a smooth domain with connected boundary, then $\partial\Omega$ lies within an annulus of thickness $C \operatorname{osc}(H)$. C depends on $|\partial\Omega|$ and a lower bound for interior and exterior balls.

Generalization to spaceforms and other curvature functions

$$F = F(\kappa_i)$$

was given by Ciraolo/Roncoroni/Vezzoni.²

²*Quantitative stability for hypersurfaces with almost constant curvature in space forms, Ann. Mat. Pura Appl. **200** (2021), no. 5, 2043–2083.*

The question of stability

Theorem (Rolando Magnanini and Giorgio Poggesi)

On the stability for Alexandrov's soap bubble theorem, J. Anal. Math. **139** (2019), no. 1, 179–205.

Let Ω be a smooth domain with connected boundary, then $\partial\Omega$ lies within an annulus of thickness at most $C\|H - H_0\|_{L^1(\partial\Omega)}^{\tau_n}$, where

$$H_0 = \frac{n}{n+1} \frac{|\partial\Omega|}{|\Omega|},$$

τ_n is a dimensional constant and C depends on few geometric quantities, such as interior and exterior ball conditions.

Ambient anisotropic geometry

- Let \mathcal{W} (aka Wulff shape) be the smooth boundary of a convex body \mathcal{W}_0 containing the origin.
 - ▶ Minkowski norm

$$F^0(x) = \inf\{s > 0: x \in s\mathcal{W}_0\}$$

- ▶ Then $\mathcal{W} = \{F^0 = 1\}$.
- $q(x) = \frac{1}{2}(F^0(x))^2$ is smooth and strictly convex, hence

$$\bar{g}(x) := D^2q(x)$$

is a metric on $\mathbb{R}^{n+1} \setminus \{0\}$.

- Let $F(z) = \sup_{x \in \mathcal{W}} \langle x, z \rangle$ be the support function of \mathcal{W} ,
 - ▶ $\Phi = (\text{grad } F)|_{\mathbb{S}^n}: \mathbb{S}^n \rightarrow \mathcal{W}$ is an embedding of the Wulff shape.
 - ▶ Φ is the inverse of the Gauss map of \mathcal{W} .

Induced anisotropic geometry

- Let $x: M^n \rightarrow \mathbb{R}^{n+1}$ embedding with M closed manifold, $\tilde{\nu}: M \rightarrow \mathbb{S}^n$ its normal vector field aka Gauss map.
 - ▶ *Anisotropic normal* $\nu = \Phi \circ \tilde{\nu} =$ “position $\nu(x)$ on the Wulff shape, at which the normal of the Wulff shape equals $\tilde{\nu}(x)$ ”
- Tangent space coincidence: $T_x M = T_{\tilde{\nu}(x)} \mathbb{S}^n = T_{\nu(x)} \mathcal{W}$.
 - ▶ Homogeneity of F^0 implies for all tangent vectors $V \in T_x M$,

$$\bar{g}_\nu(\nu, \nu) = 1, \quad g_\nu(\nu, V) = 0$$

- ▶ *Induced anisotropic metric and second fundamental form:*

$$g_{ij}(x) = \bar{g}_{\nu(x)}(\partial_i x, \partial_j x), \quad h_{ij} = \bar{g}_{\nu(x)}(\partial_i \nu, \partial_j x).$$

- ▶ *Anisotropic mean curvature* $H = g^{ij} h_{ij} \equiv \text{tr}(A)$.
- *Anisotropic volume element* $d\mu = F(\tilde{\nu}) d \text{vol}_n$ (\neq volume element induced from g).

Level sets

- For a function f on \mathbb{R}^{n+1} , define the F -gradient by

$$\nabla_F^\# f = F(D^\# f) D^\# F(D^\# f).$$

- ▶ On a regular level set M of f this coincides with the definition

$$\bar{g}_{\nu(x)}(\nabla_F^\# f, V) = Df(x)V \quad \forall V \in T_x M.$$

- Define the F -Hessian endomorphism by

$$\nabla_F^2 f = D^2\left(\frac{1}{2}F^2\right)(D^\# f) \circ D^2 f$$

- ▶ On a regular level set M of f this coincides with

$$\nabla_F^2 f = \bar{g}_{\nu(x)}^{-1} \circ D^2 f$$

- ▶ Operator degenerates where $Df = 0$.

Anisotropic level-set stability

Theorem (with Xuwen Zhang)

Stability of the Wulff shape with respect to anisotropic curvature functionals, (2023).

Let $n \geq 2$, $M \subset \mathbb{R}^{n+1}$ closed hypersurface, F an elliptic integrand, $\mu(M) = 1$. Let \mathcal{U} be one-sided neighbourhood of M , given by level sets of $f \in C^2(\bar{\mathcal{U}})$,

$$\bar{\mathcal{U}} = \bigcup_{0 \leq t \leq \max|f|} M_t, \quad M_t = \{|f| = t\},$$

with $f|_M = 0$ and $df|_{\bar{\mathcal{U}}} > 0$. Let $p > n$ and $\max_{0 \leq t \leq \max|f|} \|A\|_{p, M_t} \leq C_0$. Then

$$\text{dist}(M, \mathcal{W}) \leq \frac{C(n, p, F, C_0)}{\min(\max|f|, \min|df|)^{\frac{p}{p+1}}} \left(\int_{\mathcal{U}} |\mathring{\nabla}_F^2 f|^p \right)^{\frac{1}{p+1}}$$

for the Wulff shape corresponding to F , provided the RHS is small.

Some words about the proof

If h is the anisotropic 2^{nd} fundamental form of any regular level set of f , then

$$\nabla^2 f|_M = F(D^\sharp f)h.$$

Hence

$$F^2(D^\sharp f)|\mathring{A}|^2 \leq |\mathring{\nabla}_F^2 f|^2,$$

where \mathring{A} is the tracefree part of the anisotropic second fundamental form,

$$|\mathring{A}|^2 = c_n \sum_{i < j} (\kappa_i - \kappa_j)^2.$$

Some words about the proof

Hence \mathring{A} can be controlled by $\mathring{\nabla}_F^2 f$.

Classical result from (isotropic) hypersurface theory, due to Darboux:

$$\mathring{A} = 0 \quad \Rightarrow \quad M = \text{Sphere.}$$

- ▶ A similar result holds in the anisotropic world³

Stability versions in the isotropic case are plentyfull and due to De Lellis/Müller, Topping, Grosjean.

- ▶ In the anisotropic world there is a recent one...

³Yijun He and Haizhong Li: *Integral formula of Minkowski type and new characterization of the Wulff shape*, Acta Math. Sin. **24** (2008), no. 4, 697–704.

Some words about the proof

Theorem (Antonio De Rosa, Stefano Giofré)

Absence of bubbling phenomena for non-convex anisotropic nearly umbilical and quasi-Einstein hypersurfaces, J. Reine Angew. Math. **780** (2021), 1–40.

Let $M \subset \mathbb{R}^{n+1}$ be a closed hypersurface, \mathcal{W} a Wulff shape, $p > n$ and

$$|M| = |\mathcal{W}|, \quad \|A\|_{L^p(M)} \leq c_0.$$

Then there exist $C = C(n, p, F, c_0) > 0$, such that: if

$$\|\dot{A}\|_{L^p(M)} \leq \frac{1}{C},$$

then there exists $c \in \mathbb{R}^{n+1}$ and a parametrization $\psi: \mathcal{W} \rightarrow M$, such that

$$\|\psi - \text{id} - c\|_{W^{2,p}(\mathcal{W})} \leq C \|\dot{A}\|_{L^p(M)}.$$

Application I: Anisotropic Heintze-Karcher

- **Stability of the domain in the Heintze-Karcher inequality.** In the Euclidean space \mathbb{R}^{n+1} , for every domain Ω with mean-convex $\partial\Omega$:

$$\int_{\partial\Omega} \frac{n}{H} \geq (n+1) \text{vol}(\Omega)$$

with equality precisely when Ω is a ball.

- ▶ The same holds in the anisotropic setting, if we integrate w.r.t. the anisotropic area measure.

Application I: Anisotropic Heintze-Karcher

Theorem (with Xuwen Zhang, Stability in the anisotropic Heintze-Karcher)

Let $n \geq 2$, $\alpha > 0$ and F be an elliptic integrand on \mathbb{R}^{n+1} . Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain with connected F -mean convex $C^{2,\alpha}$ -boundary that satisfies a uniform interior Wulff sphere condition with radius r . Then there exists a positive constant C depending only on $n, \alpha, F, r, \mu(\partial\Omega)$ and $|\partial\Omega|_{2,\alpha}$, such that

$$\text{dist}(\partial\Omega, \mathcal{W}) \leq C \left(\int_{\partial\Omega} \frac{1}{H} d\mu - \frac{n+1}{n} |\Omega| \right)^{\frac{1}{n+2}}$$

for some Wulff sphere $\mathcal{W} \subset \mathbb{R}^{n+1}$, provided the RHS is sufficiently small.

Proof of Heintze-Karcher stability

- The key for stability is the following estimate

$$\int_{\Omega} |\mathring{\nabla}_F^2 f|^2 dx \leq \left(\frac{n}{n+1} \right)^2 \int_{\partial\Omega} \frac{1}{H} d\mu - \frac{n+1}{n} |\Omega|,$$

where

$$\begin{aligned} \operatorname{div}(D^\sharp(\tfrac{1}{2}F^2)(D^\sharp f)) &=: \Delta_F f = 1 \quad \text{in } \Omega \\ f &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

- ▶ Follows from divergence theorem type argument and Hölder's inequality.
- ▶ f shall serve as the foliation function in a neighbourhood of $\partial\Omega$.
- ▶ For this we need a lower gradient bound of f on $\partial\Omega$.

Proof of Heintze-Karcher stability

Lemma (Gradient bound on $\partial\Omega$)

Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain with C^2 -boundary that satisfies the uniform interior Wulff sphere condition with radius r and let $f \in C^{1,\beta}(\bar{\Omega}) \cap W^{2,2}(\Omega)$ for some $\beta \in (0, 1)$ be a solution of

$$\begin{aligned} \operatorname{div}(D^\sharp(\tfrac{1}{2}F^2)(D^\sharp f)) &= 1 && \text{in } \Omega \\ f &= 0 && \text{on } \partial\Omega. \end{aligned}$$

then

$$|D^\sharp f| \geq C(n, F)r \quad \text{on } \partial\Omega.$$

- From here, higher regularity in a controlled neighbourhood of $\partial\Omega$ follows from Schauder theory.
- The level set stability theorem completes the proof of the anisotropic Heintze-Karcher stability.

Application II: Stability in the anisotropic soap bubble theorem

Theorem (with Xuwen Zhang, Stability in the anisotropic soap bubble theorem)

Let $n \geq 2$, $\alpha > 0$ and F be an elliptic integrand on \mathbb{R}^{n+1} . Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain with connected boundary $\partial\Omega \in C^{2,\alpha}$ that satisfies a uniform interior Wulff sphere condition with radius r . Then there exists a positive constant C depending only on $n, \alpha, F, r, \mu(\partial\Omega)$, such that

$$\text{dist}(\partial\Omega, \mathcal{W}) \leq C \left\| H - \frac{n}{n+1} \frac{\mu(\partial\Omega)}{|\Omega|} \right\|_{1, \partial\Omega}^{\frac{1}{n+2}}$$

for some Wulff sphere $\mathcal{W} \subset \mathbb{R}^{n+1}$, provided the RHS is sufficiently small.

Integral approach

Key ingredient in the isotropic case (A. Ros): Use **Reilly's integral identity**.

For C^2 -functions f on a domain $\Omega \subset \mathbb{R}^{n+1}$, with $f|_{\partial\Omega} = \text{const}$:

$$\int_{\Omega} (\Delta f)^2 - \int_{\Omega} |\nabla^2 f|^2 = \int_{\partial\Omega} H(\partial_{\nu} f)^2.$$

Cauchy-Schwarz-deficit:

$$|\overset{\circ}{\nabla}^2 f|^2 = |\nabla^2 f|^2 - \frac{1}{n+1}(\Delta f)^2,$$

Then

$$\int_{\Omega} |\overset{\circ}{\nabla}^2 f|^2 = \frac{n}{n+1} \int_{\Omega} (\Delta f)^2 - \int_{\partial\Omega} H(\partial_{\nu} f)^2.$$

Solve

$$\begin{aligned} \Delta f &= 1 && \text{in } \Omega \\ f &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Then

$$\begin{aligned}\int_{\Omega} |\dot{\nabla}^2 f|^2 &= \frac{n}{(n+1) \text{vol}(\Omega)} \left(\int_{\Omega} \Delta f \right)^2 - \int_{\partial\Omega} H(\partial_{\nu} f)^2 \\ &= \frac{n}{(n+1) \text{vol}(\Omega)} \left(\int_{\partial\Omega} \partial_{\nu} f \right)^2 - \int_{\partial\Omega} H(\partial_{\nu} f)^2 \\ &\leq \frac{n}{(n+1)} \frac{\text{Area}(\partial\Omega)}{\text{vol}(\Omega)} \int_{\partial\Omega} (\partial_{\nu} f)^2 - \int_{\partial\Omega} H(\partial_{\nu} f)^2 \\ &\equiv \int_{\partial\Omega} (H_0 - H)(\partial_{\nu} f)^2.\end{aligned}$$

Application III: Serrin' problem

- Serrin's overdetermined problem asks which domains Ω allow solutions to

$$\begin{aligned}\Delta u &= 1 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \\ |\nabla u| &= c && \text{on } \partial\Omega.\end{aligned}$$

- ▶ The answer is: Only balls.
- The anisotropic version (yielding equality to the Wulff shape)

$$\begin{aligned}\operatorname{div}(D^\sharp(\tfrac{1}{2}F^2)(D^\sharp f)) &= 1 && \text{in } \Omega \\ f &= 0 && \text{on } \partial\Omega \\ F(D^\sharp f) &= c && \text{on } \partial\Omega\end{aligned}$$

is due to Cianchi/Salani.⁴

⁴*Overdetermined anisotropic elliptic problems*, Math. Ann. **345** (2009), no. 4, 859–881.

Application III: Stability in the anisotropic Serrin problem

Theorem (with Xuwen Zhang, Stability in the anisotropic Serrin problem)

Let $n \geq 2$, $\alpha > 0$ and F be an elliptic integrand on \mathbb{R}^{n+1} . Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain with connected boundary $\partial\Omega \in C^{2,\alpha}$ that satisfies a uniform interior Wulff sphere condition with radius r . Then there exists a positive constant depending only on $n, \alpha, F, r, \text{diam}(\Omega), \mu(\partial\Omega)$ and $|\partial\Omega|_{2,\alpha}$, such that

$$\text{dist}(\partial\Omega, \mathcal{W}) \leq C \left\| F(D^\# f) - \frac{|\Omega|}{\mu(\partial\Omega)} \right\|_{1, \partial\Omega}^{\frac{1}{2(n+2)}}$$

for some Wulff sphere $\mathcal{W} \subset \mathbb{R}^{n+1}$, provided the RHS is sufficiently small. Here

$$\begin{aligned} \text{div}(D^\#(\frac{1}{2}F^2)(D^\# f)) &= 1 && \text{in } \Omega \\ f &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Application III: Stability in the anisotropic Serrin problem

- The proof works via the use of a so-called P -function. In our setting,

$$P = \frac{F^2(D^\sharp f)}{2} - \frac{1}{n+1}f.$$

- ▶ A computation gives

$$\operatorname{div}(\nabla_F^\sharp P) = |\mathring{\nabla}_F^2 f|^2.$$

- The next crucial ingredient is the Pohozaev-type identity

$$\int_{\Omega} P = \frac{1}{2(n+1)} \int_{\partial\Omega} F^2(D^\sharp f) \langle x, \tilde{\nu} \rangle.$$

- Further computations lead to

$$\int_{\Omega} (-f) |\mathring{\nabla}_F^2 f|^2 dx = \frac{1}{2} \int_M \left(F^2(D^\sharp f) - \frac{|\Omega|^2}{\mu(M)^2} \right) \langle \bar{\nabla}_F^\sharp f - \bar{\nabla}_F^\sharp \ell, \tilde{\nu} \rangle d\tilde{\mu},$$

where $\ell(x) = (F^0(x))^2 / (2(n+1))$.

Thank you!